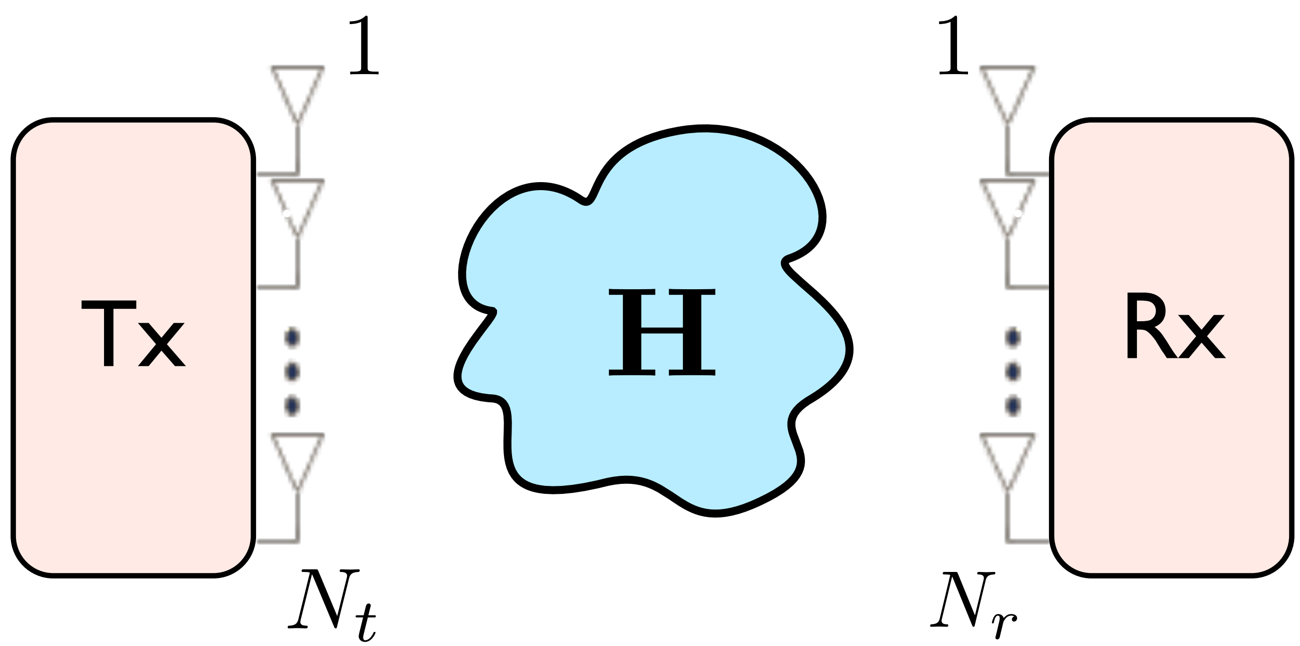
Sub-modularity and Antenna Selection in MIMO systems

Rahul Vaze
Harish Ganapathy

Point-to-Point MIMO Channel



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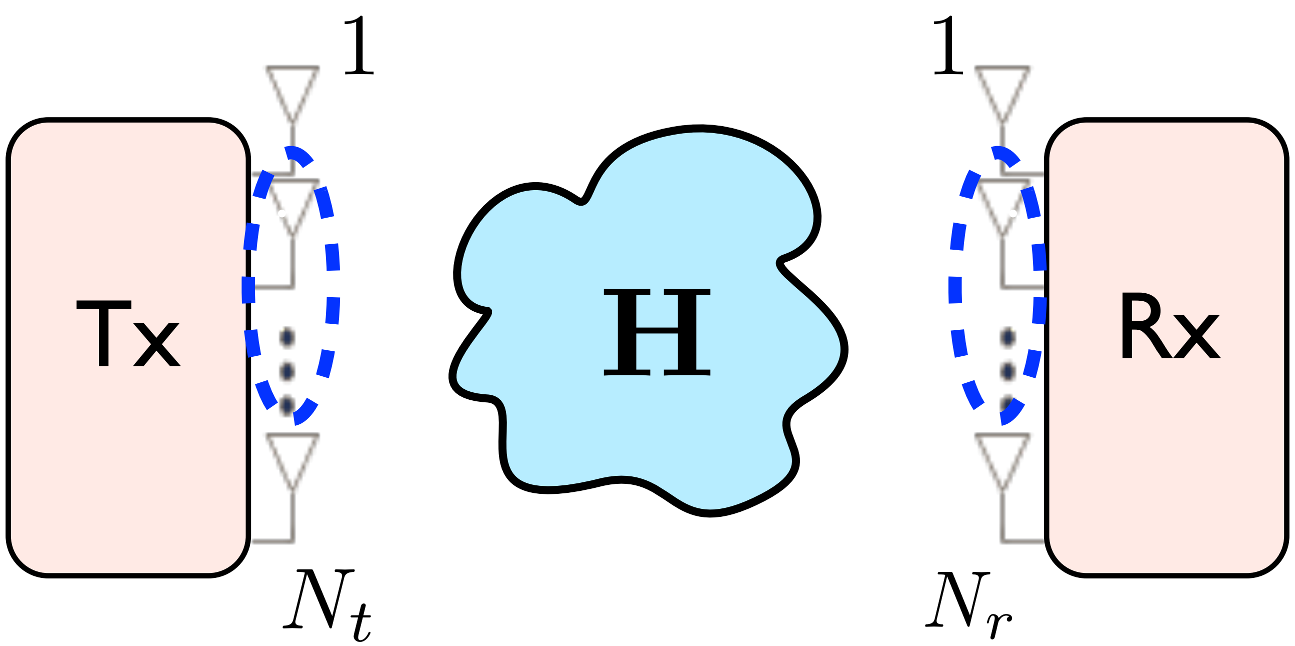

Antenna Selection

- Transmit Side
- Receive Side

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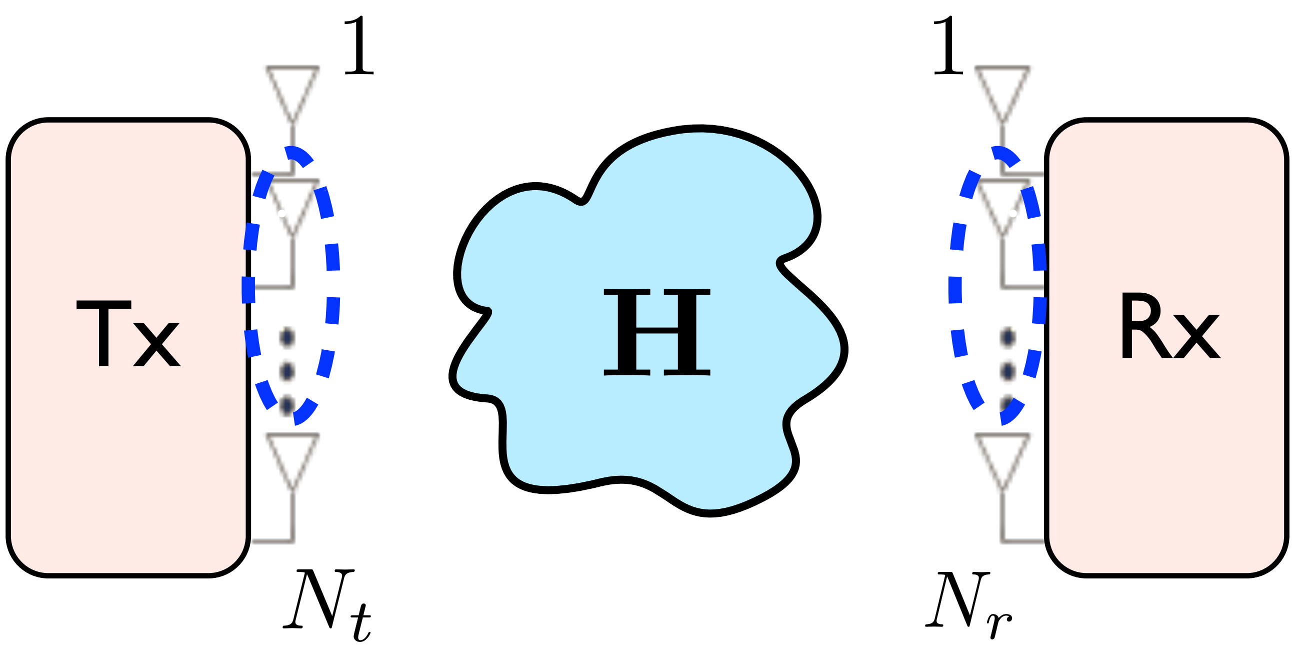

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- Mutual Information
- Reliability

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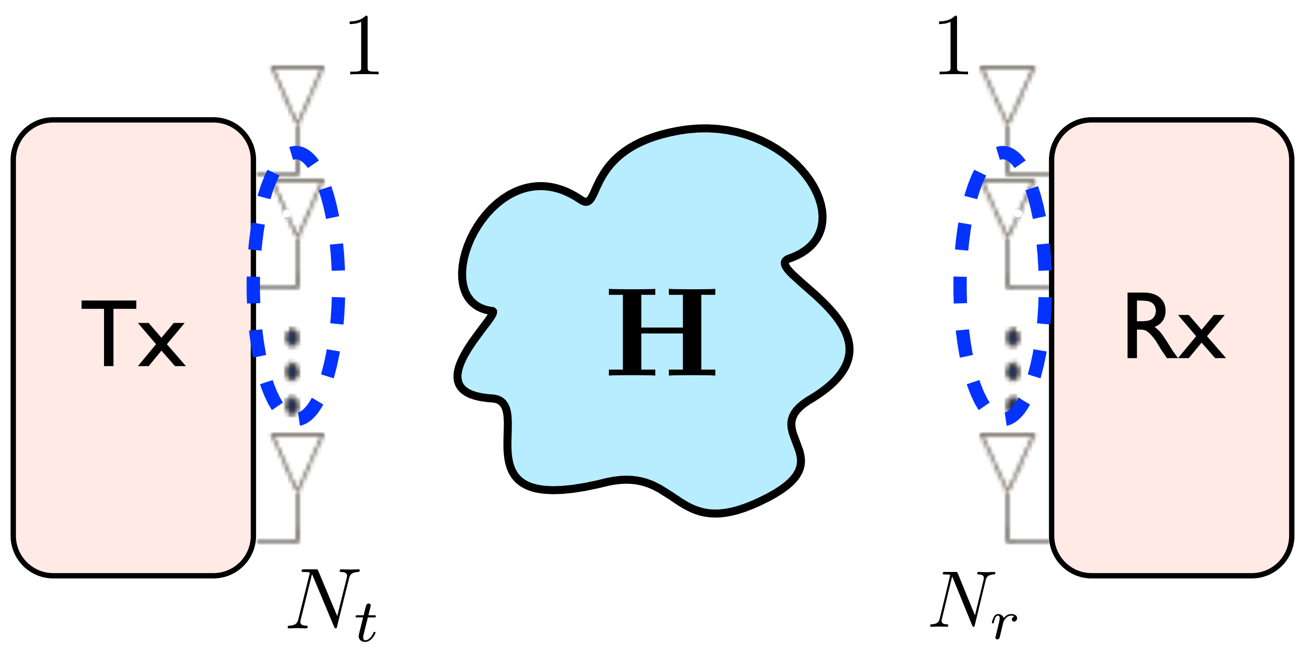

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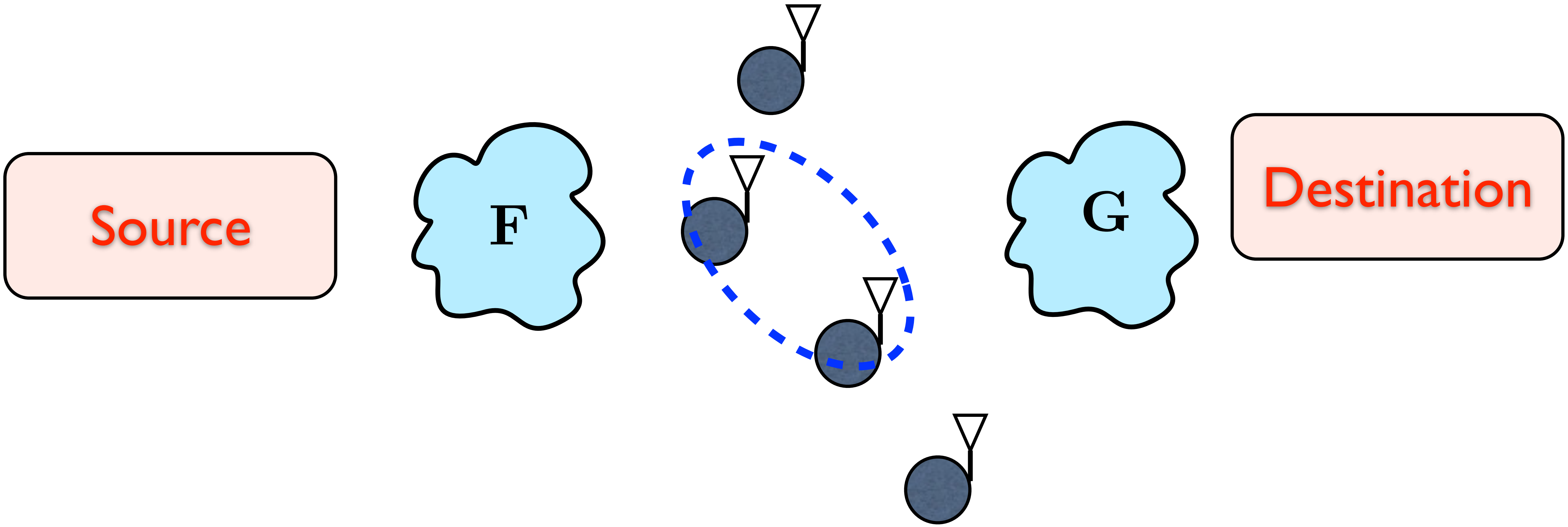
Advantages

- Simplified Circuitry
- Fewer Tx/Rx Chains

Relay Selection



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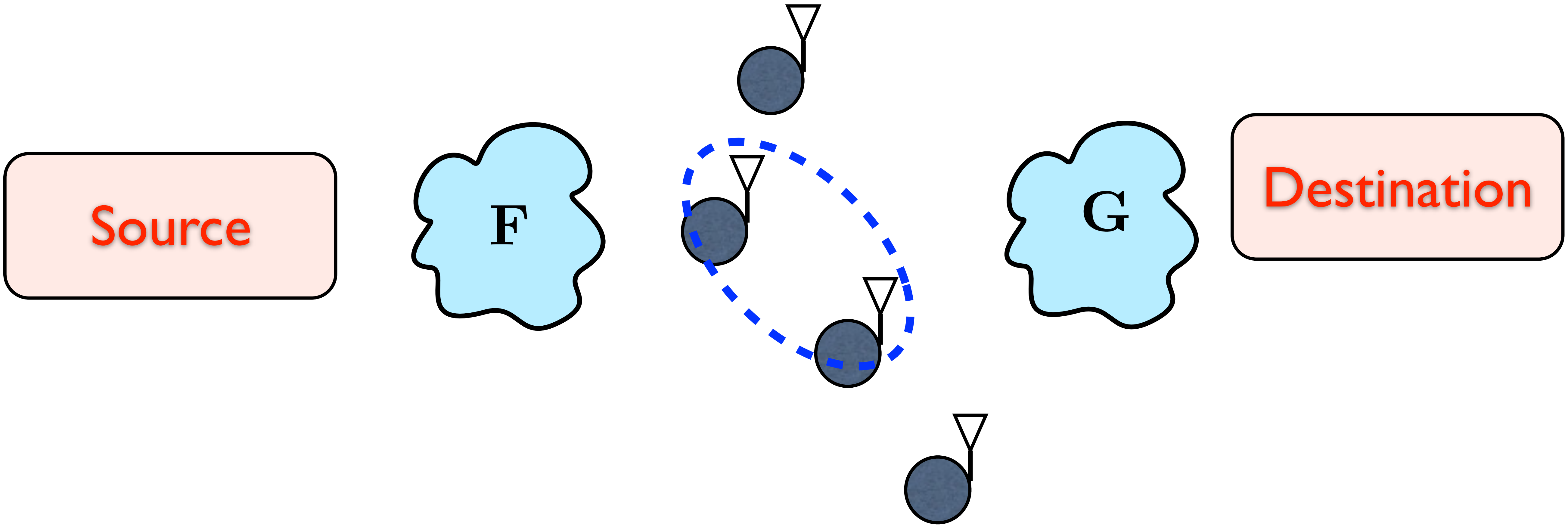

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- Amplify-forward
- Decode-forward

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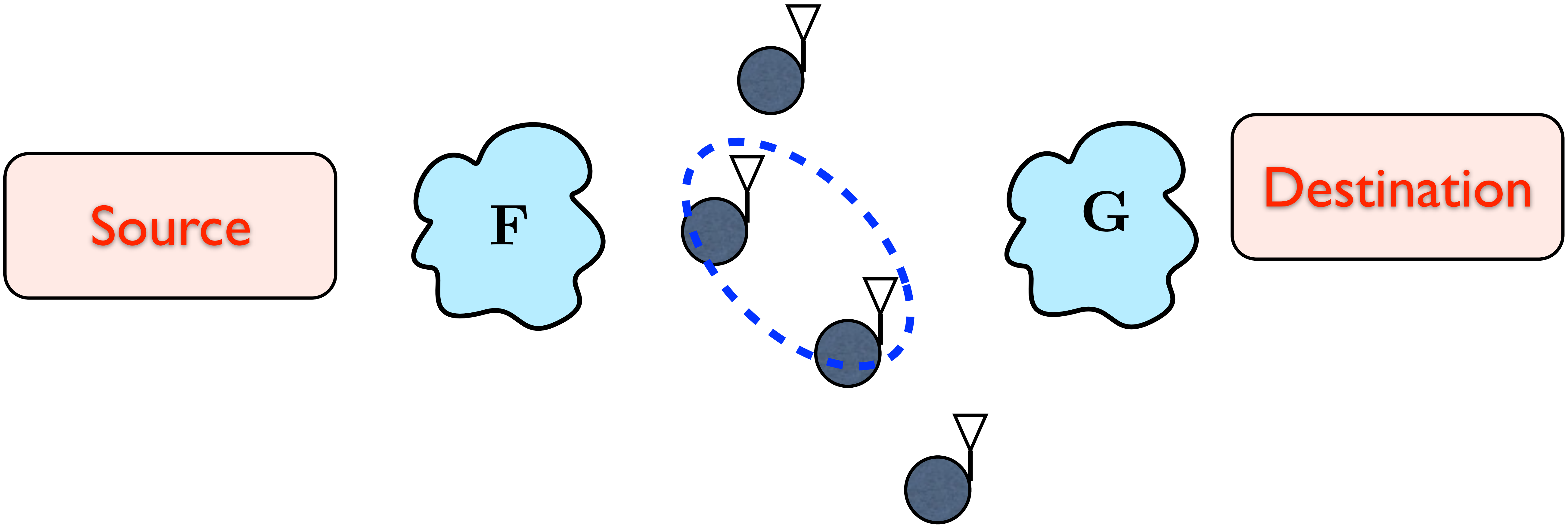

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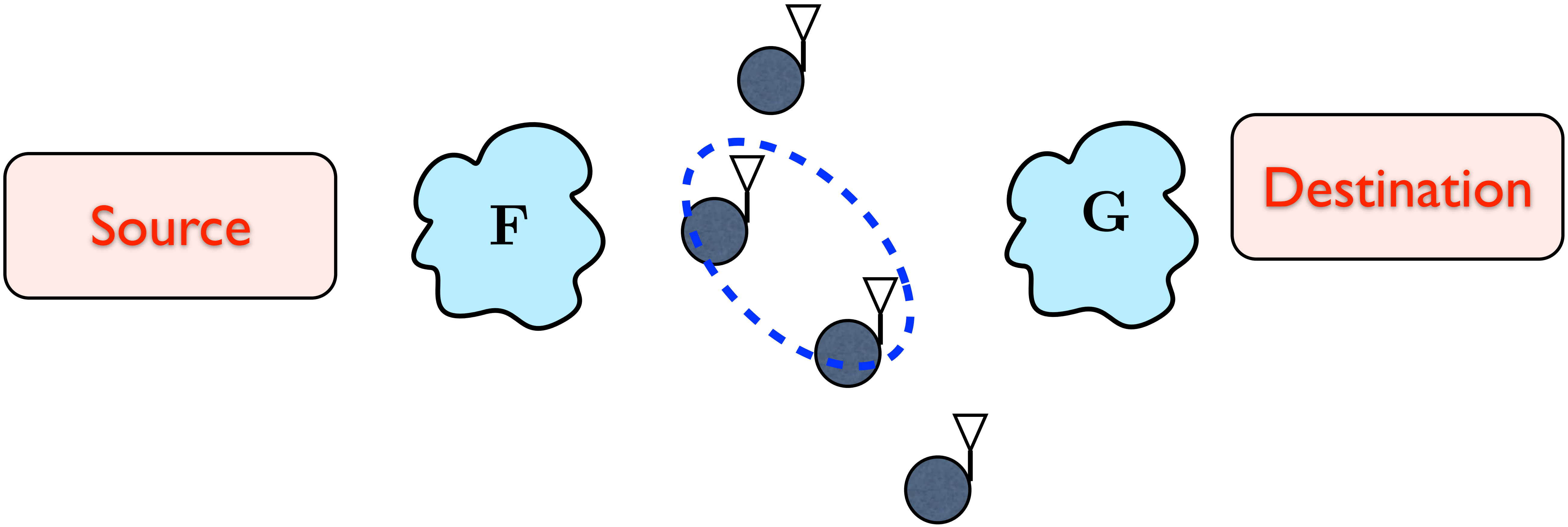

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Implementation
- Centralized
- Distributed

Prior Work

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P2P

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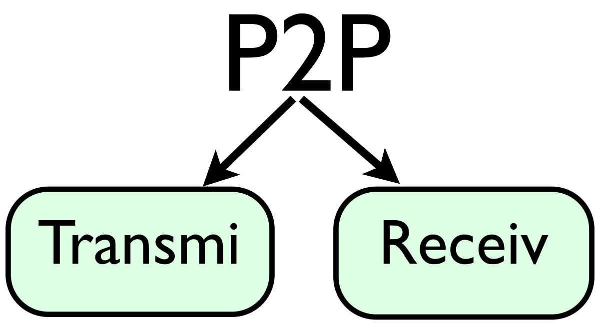

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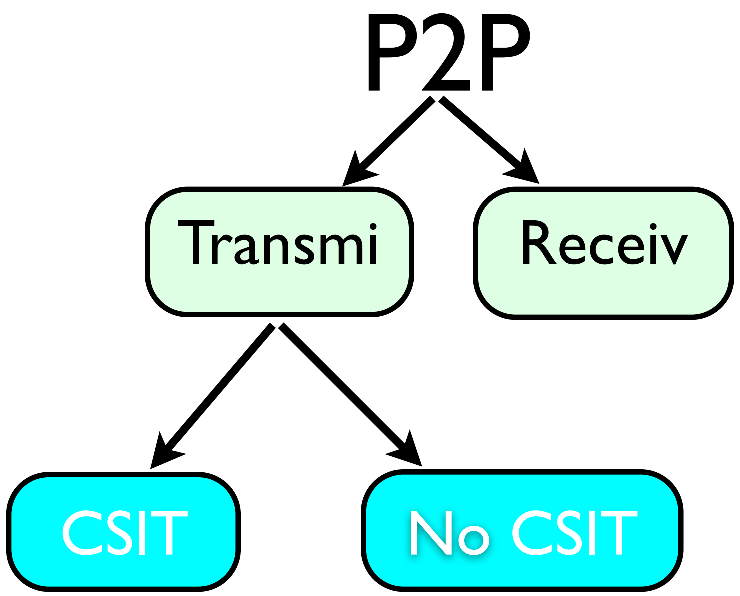

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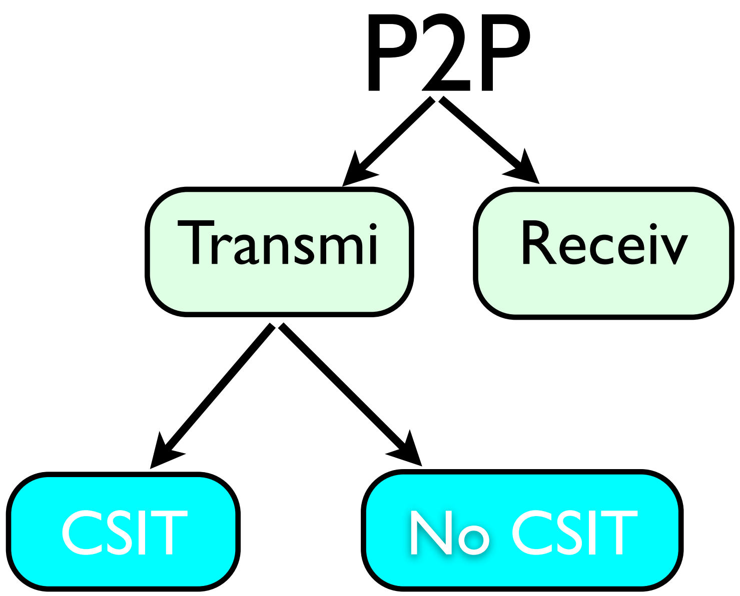

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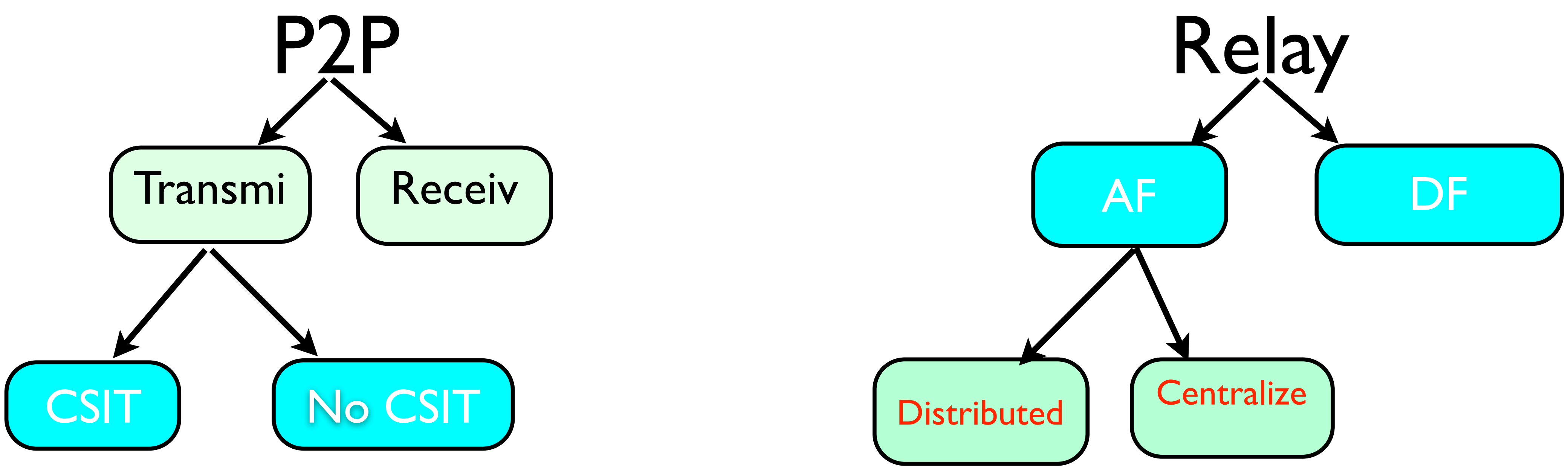

P2P
Transmi
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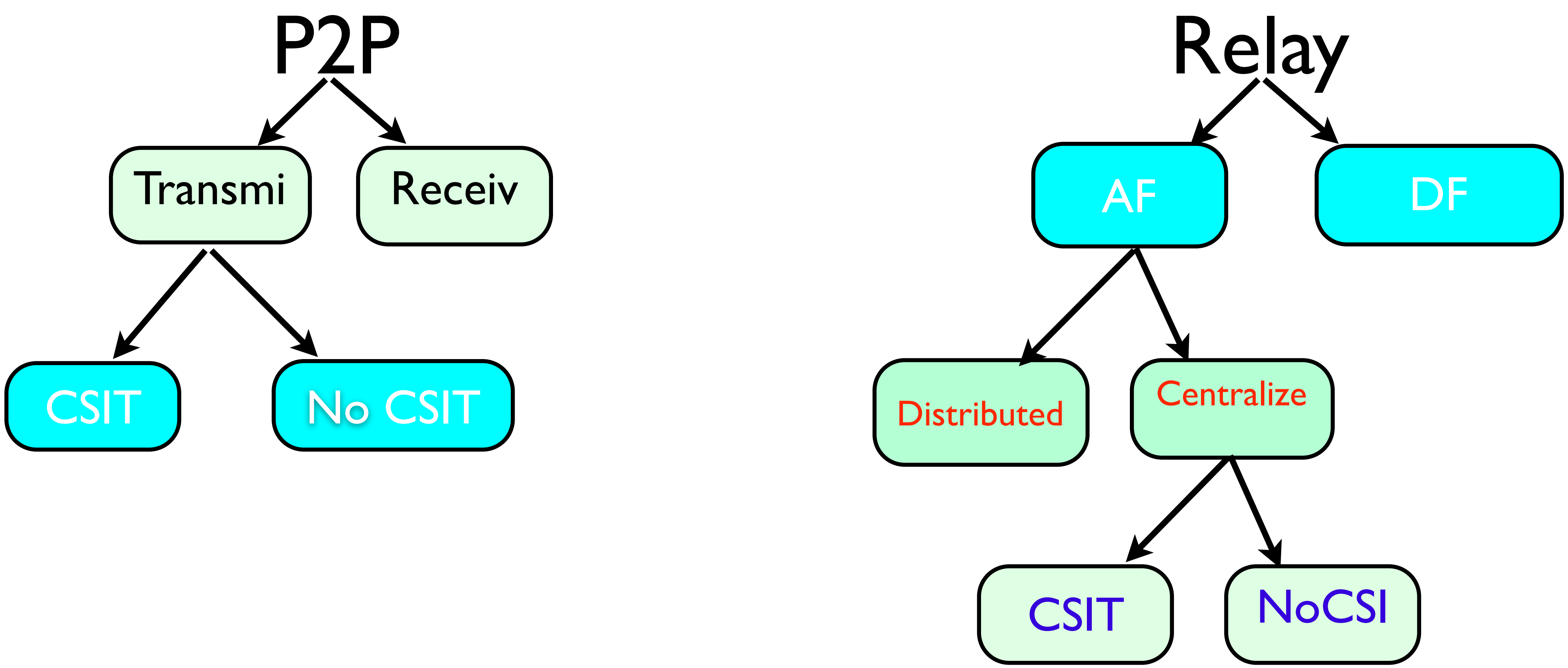
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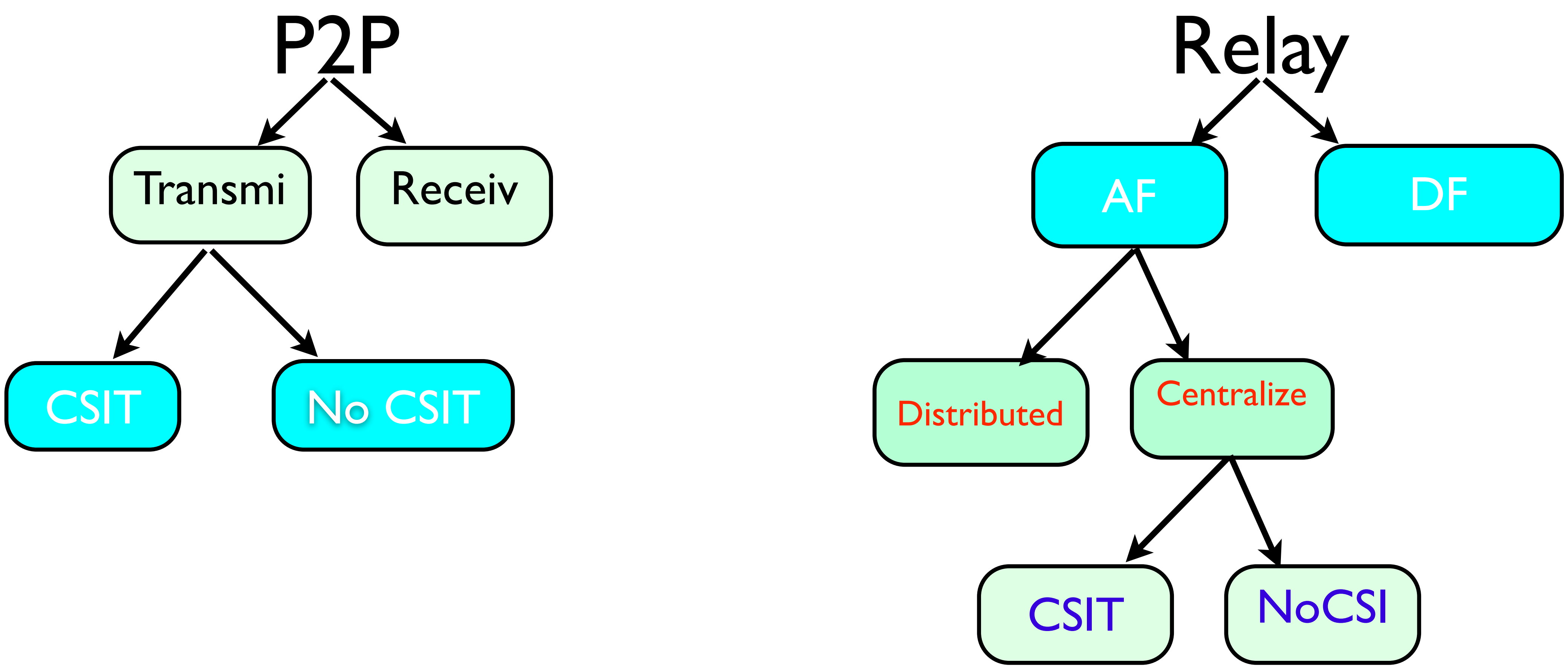

Lot of work assuming Genie-Aided Antenna Selection

No provably good simple algorithm for Antenna Selection

Implementation

- Most analytical work assumes brute-force
  (exponential complexity)

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  - If subset size is L then number of computations $\binom{N_t}{L}$

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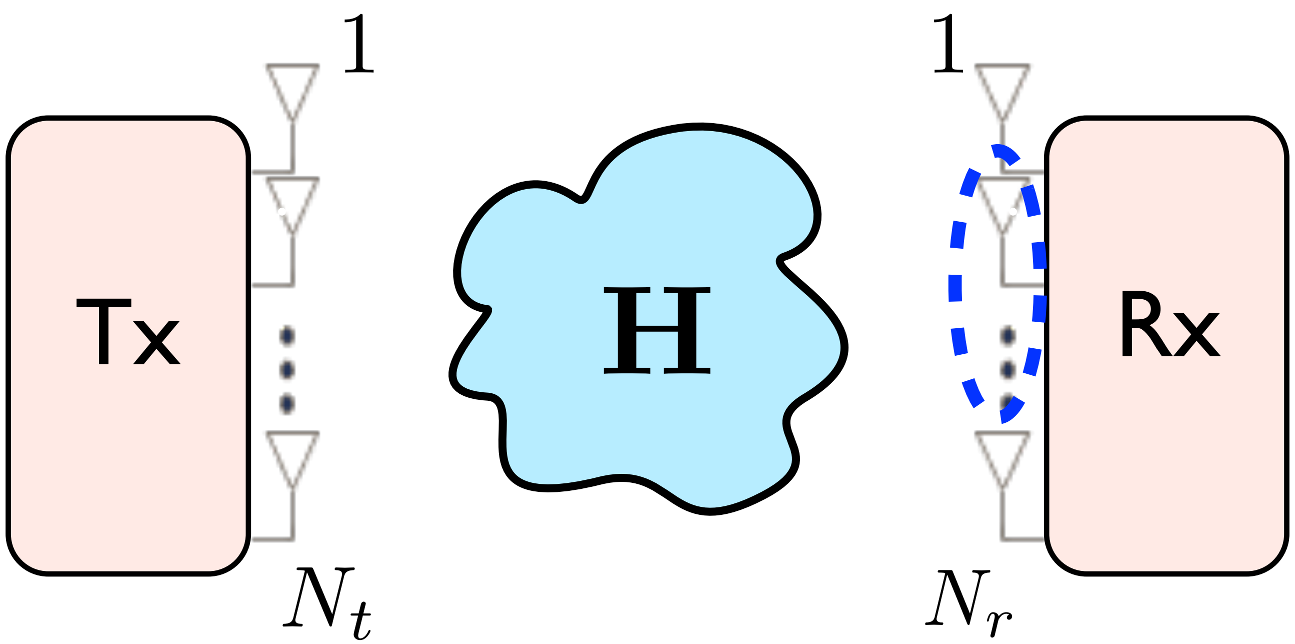
- Lots of greedy/heuristic algorithms

  No theoretical guarantees

Objective for Point-to-Point MIMO Channel



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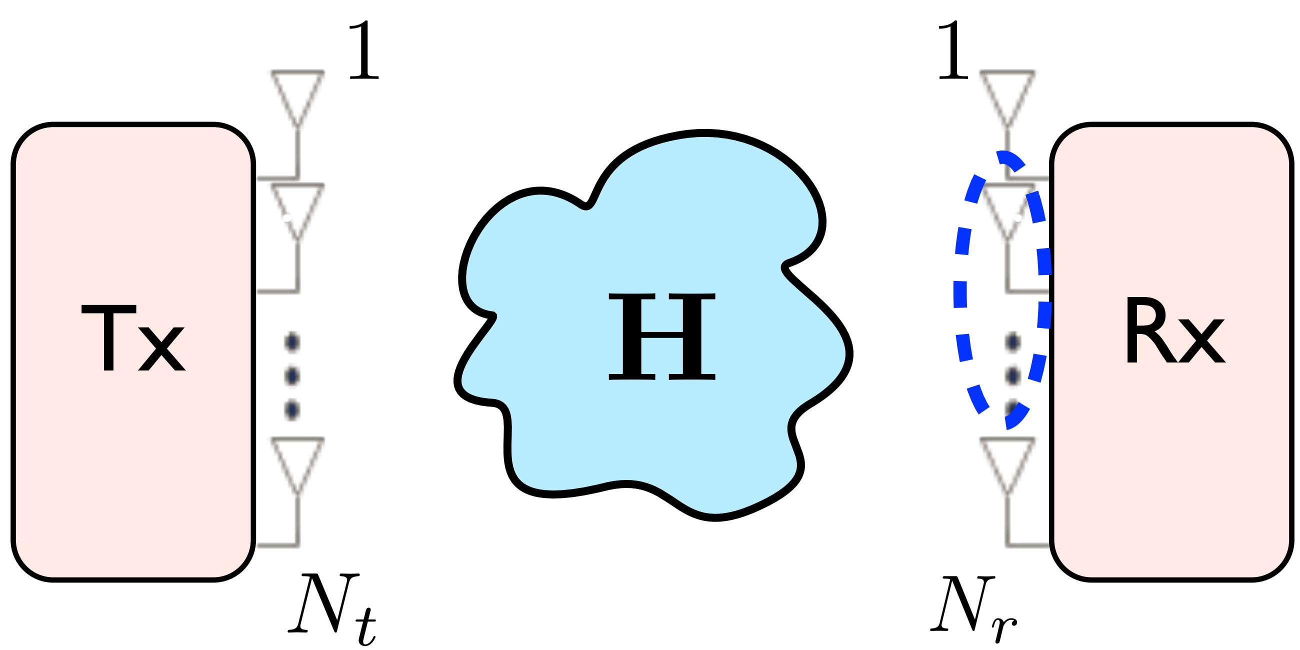
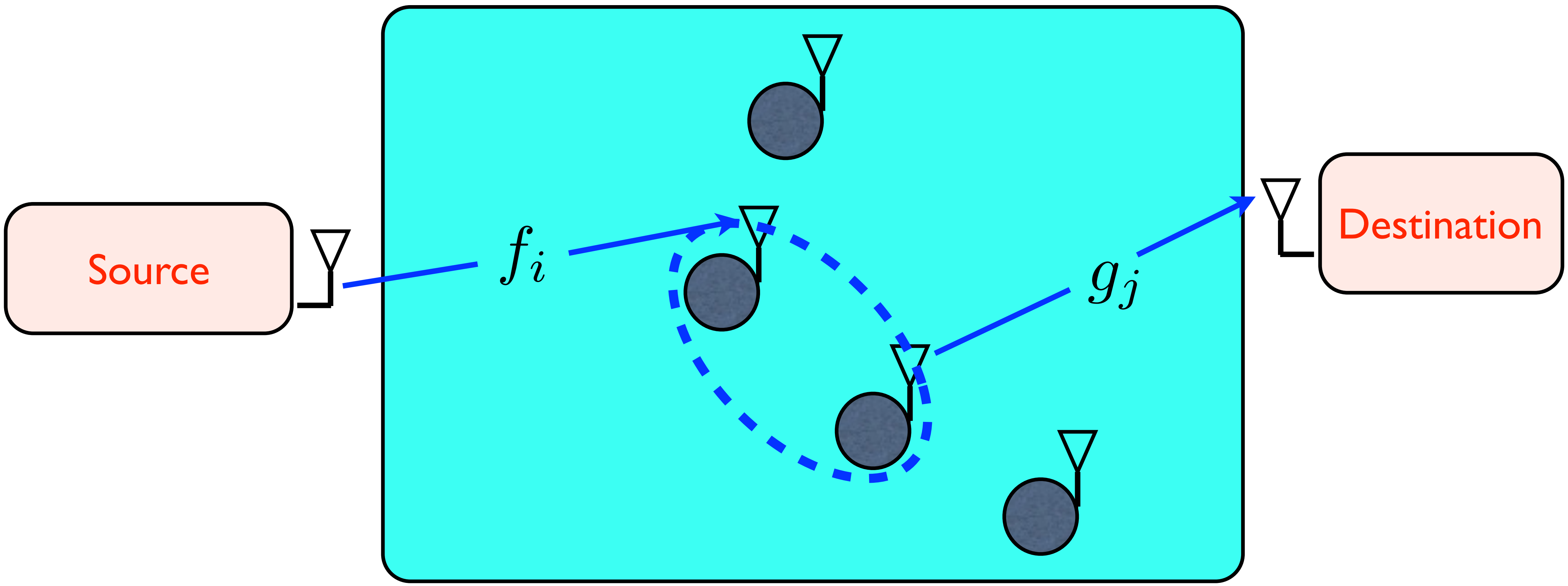


Find the size *L* receive antenna subset that maximizes the mutual information

$$\max_{\mathcal{R}_L \subset \{1,2,\ldots,N_r\}, |\mathcal{R}_L| = L} \log\det\left(\mathbf{I} + \frac{P}{N_t}\mathbf{H}_{\mathcal{R}_L}\mathbf{H}^{\dagger}_{\mathcal{R}_L}\right)$$

Objective for Relay Selection



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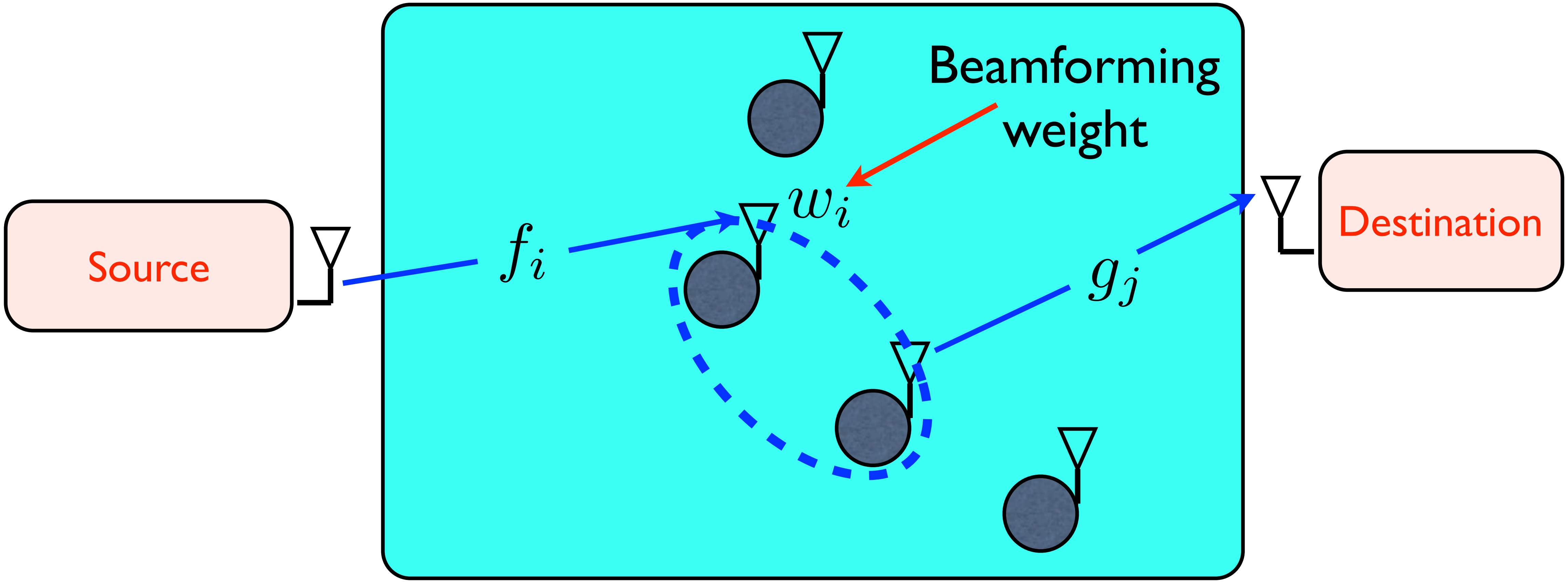

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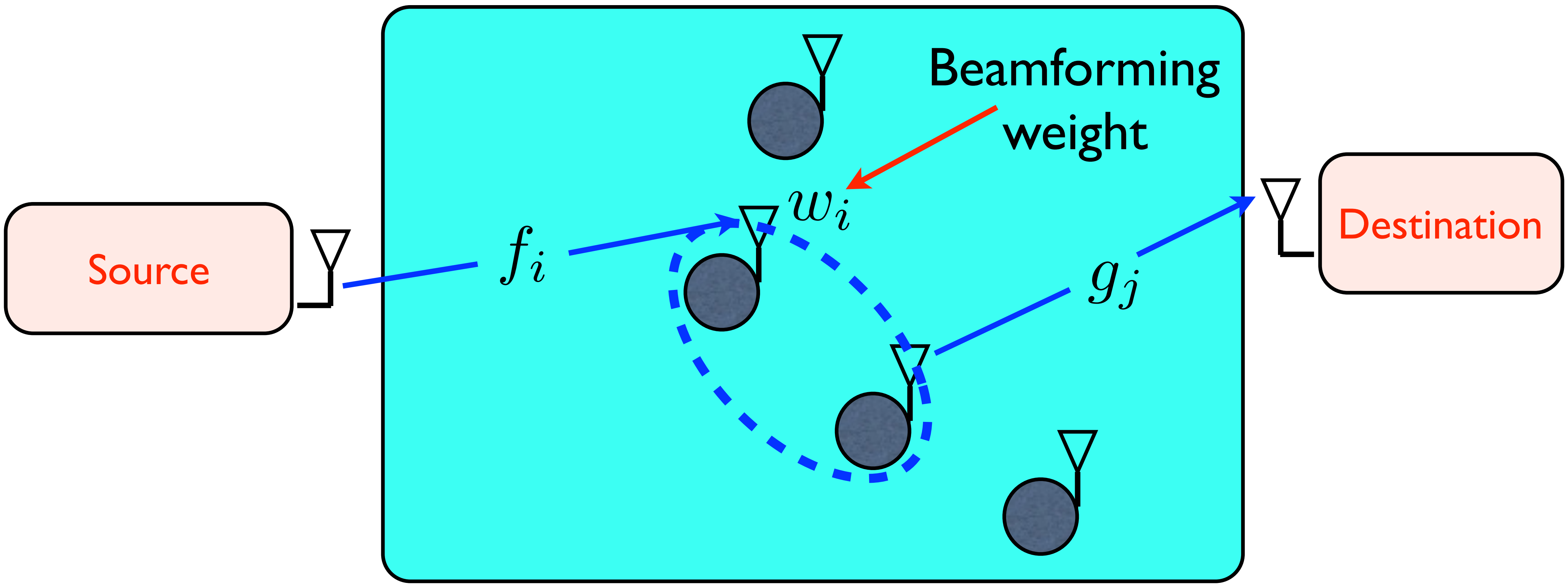

Find the size *L* relay antennas subset that maximizes the mutual information

$$\max_{\mathcal{T}_L \subseteq \{1,2,\ldots,N\}} \max_{\mathbf{w}} \log\left(1 + \frac{\mathbf{w}^\dagger \Delta \Delta^\dagger \mathbf{w}}{\mathbf{w}^\dagger (\Sigma\Sigma^\dagger + \mathbf{I})\mathbf{w}}\right)$$

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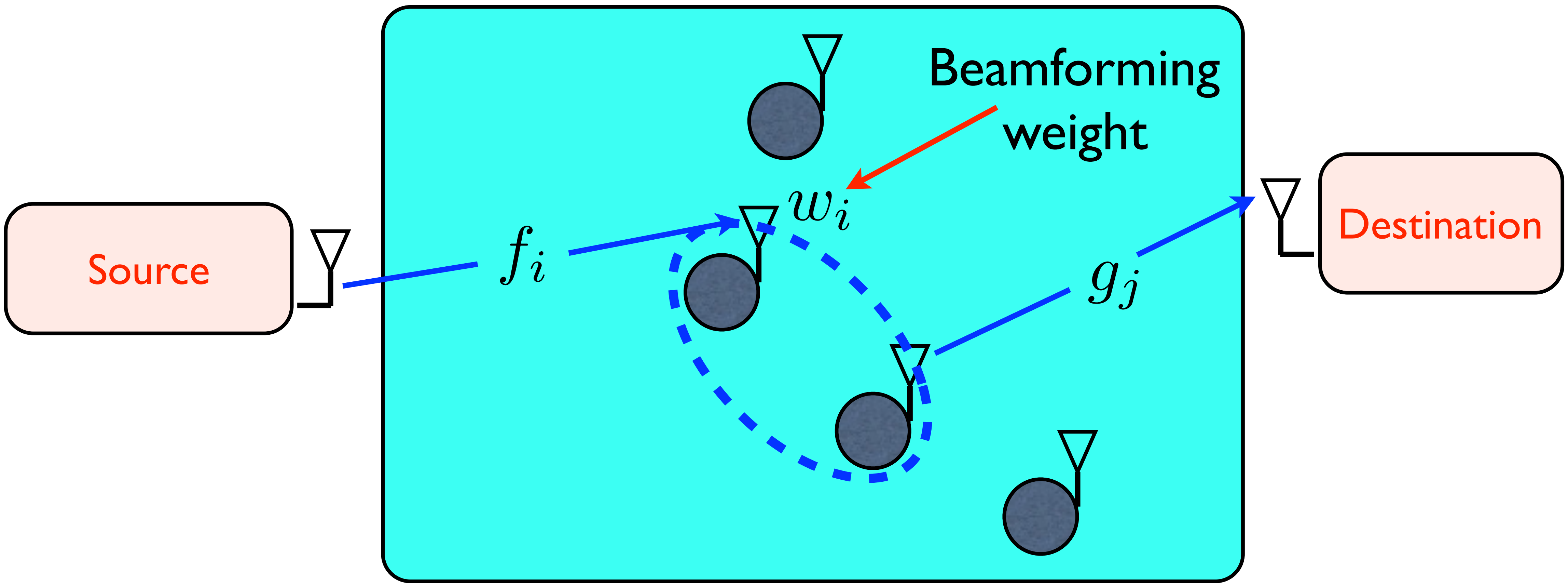

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$$\Delta = \left[\frac{g_{t_1} f_{t_1}}{\gamma_{t_1}}, \ldots, \frac{g_{t_L} f_{t_L}}{\gamma_{t_L}}\right]^T,$$

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Relay

Greedy Algorithm with linear complexity achieves the optimal solution

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Why Sub-Modular Functions ?

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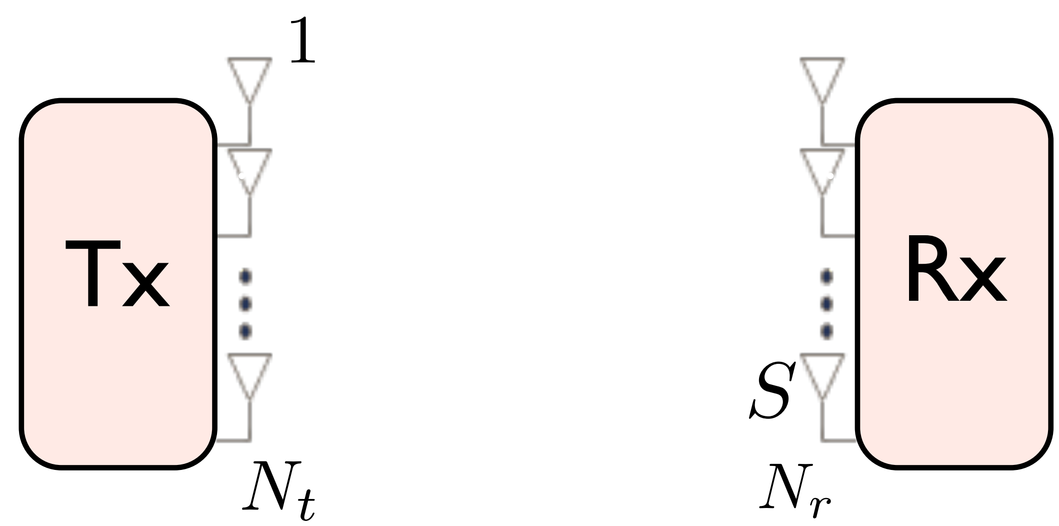
Theorem (Rado1968, Edmonds1971): If f is monotone and modular, then the greedy method achieves the optimal solution.

Receive Antenna Selection is Sub-Modular



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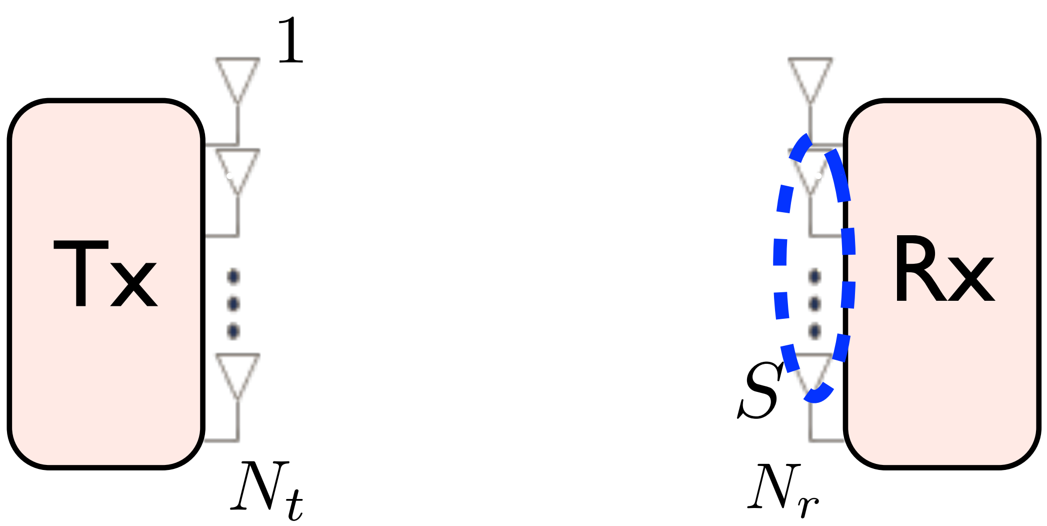

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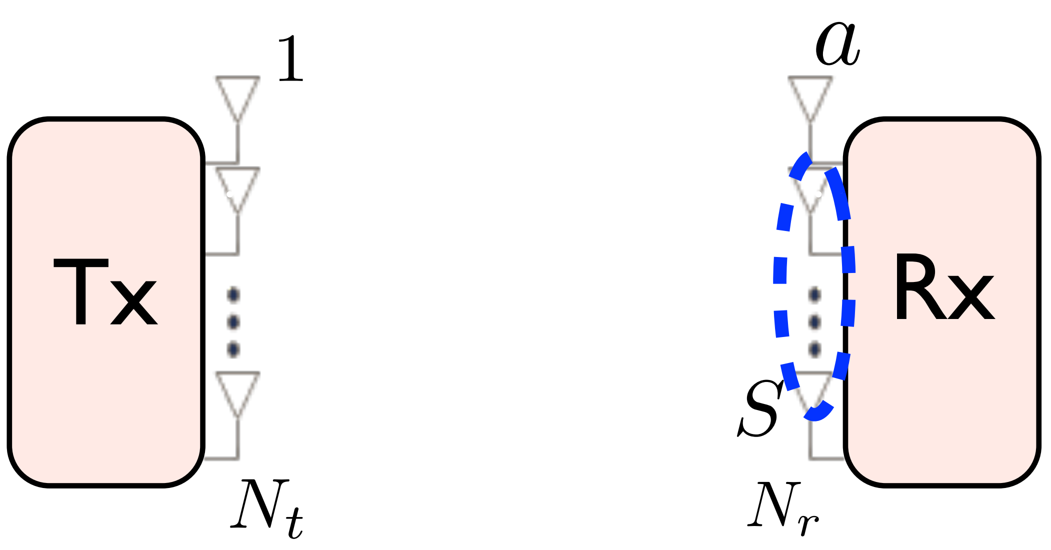

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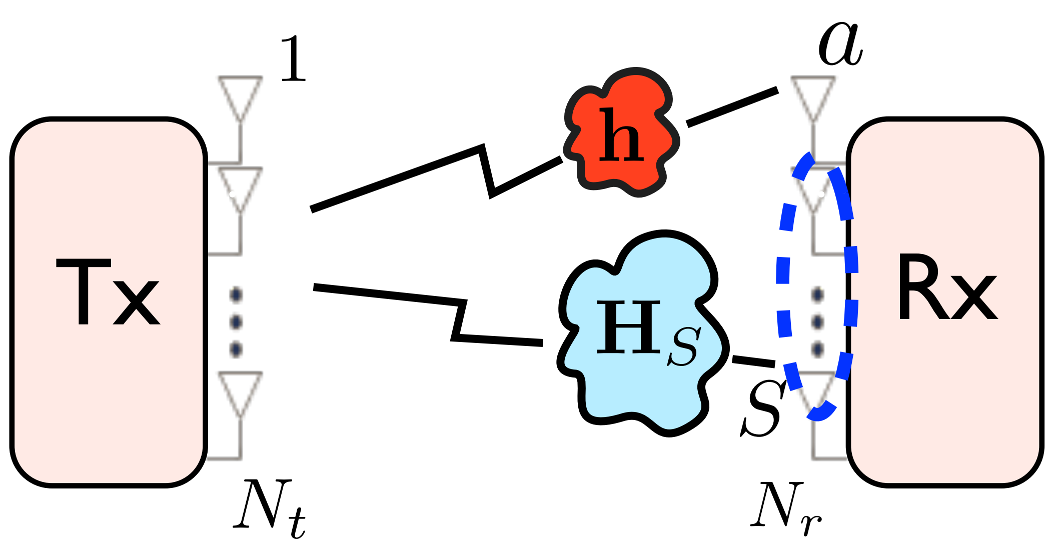

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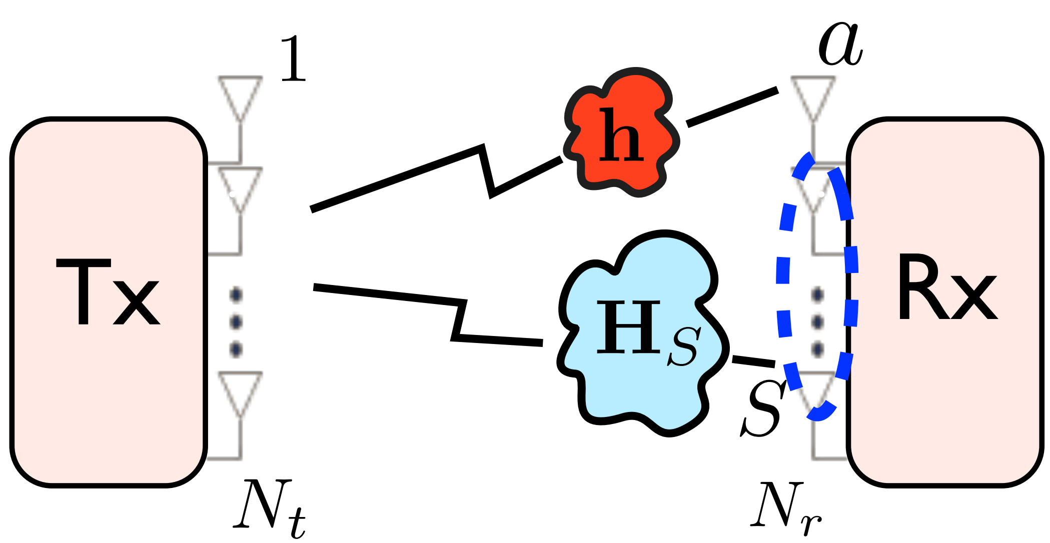

$$f(S \cup \{a\}) - f(S) = \log\det\left(\mathbf{I}_{|S|+1} + \frac{P}{N_t}\left[\begin{array}{c}\mathbf{H}_S\\ \mathbf{h}\end{array}\right][\mathbf{H}_S^\dagger \;\; \mathbf{h}^\dagger]\right) - \log\det\left(\mathbf{I}_{|S|} + \frac{P}{N_t}\mathbf{H}_S\;\mathbf{H}_S^\dagger\right),$$

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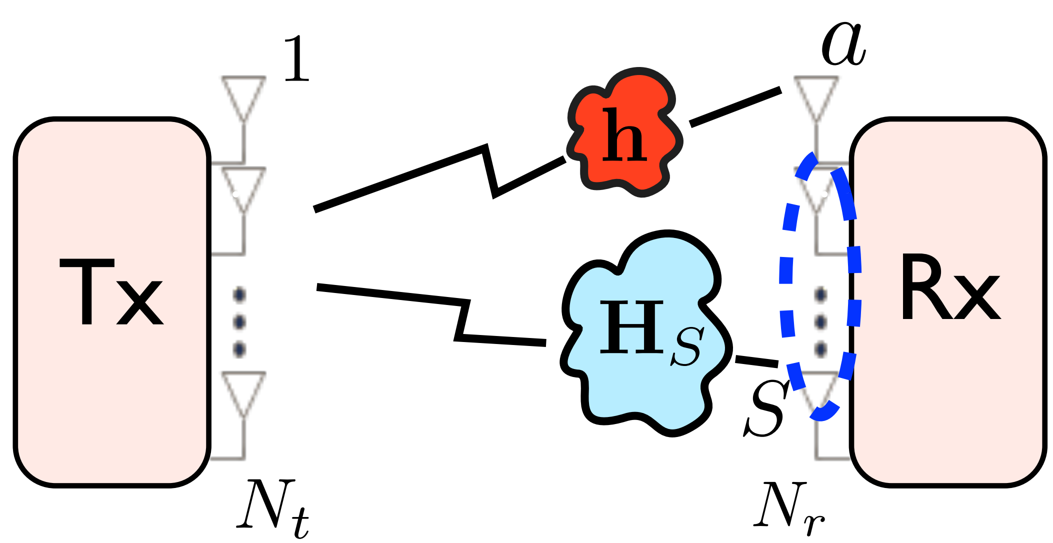

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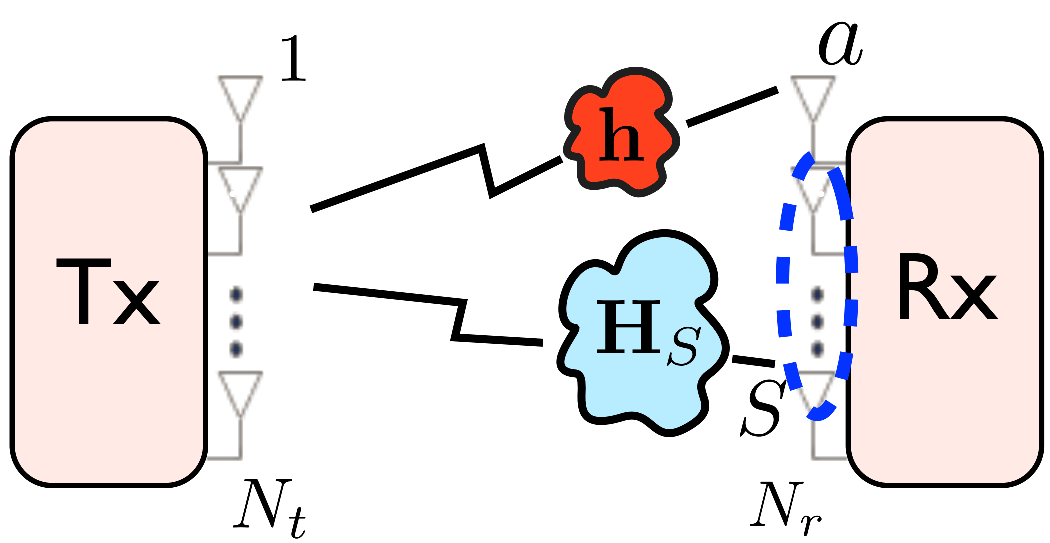

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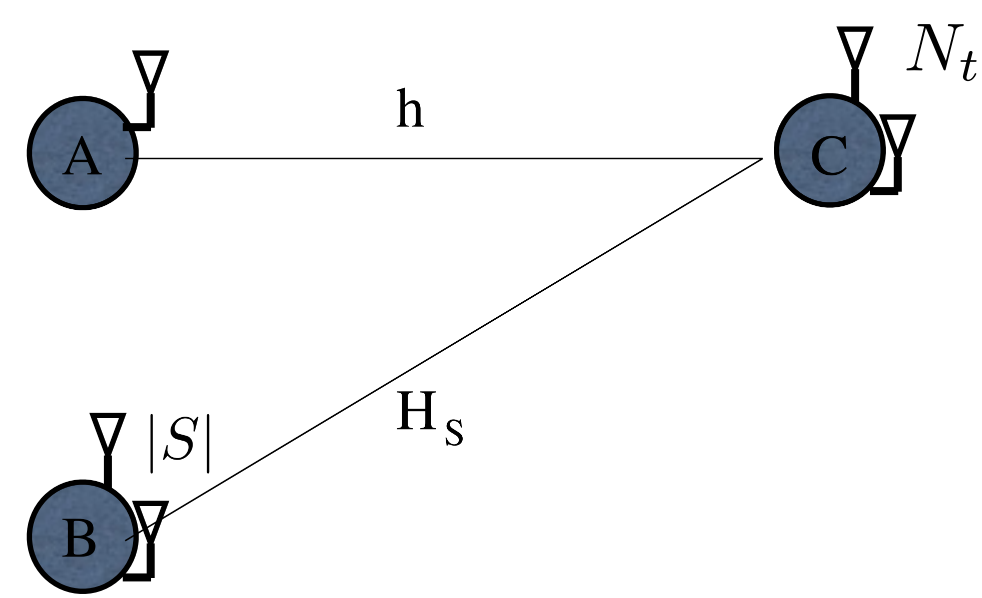
$f(S\cup\{a\}) - f(S)$

Mutual Information between A and C with Gaussian Signalling



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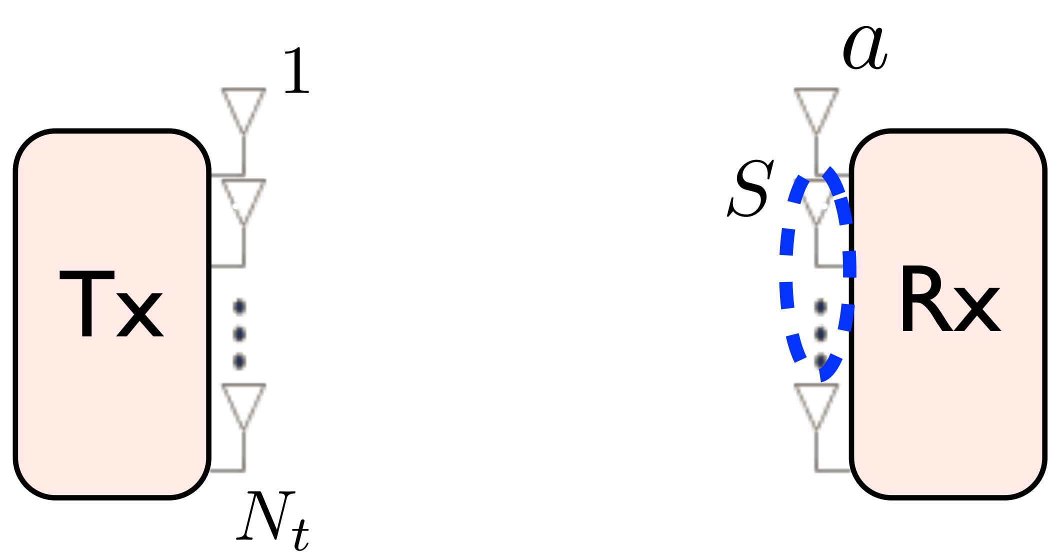

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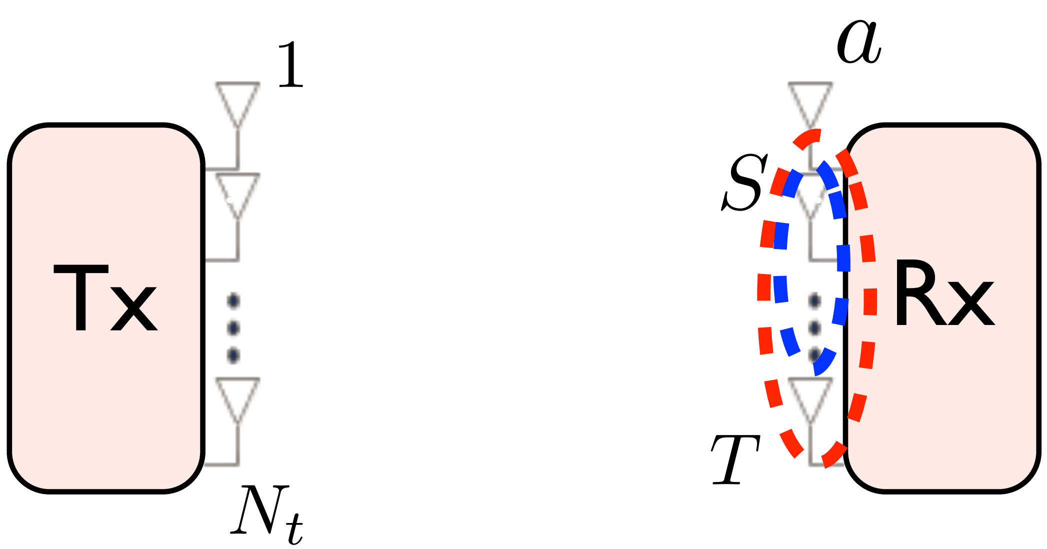

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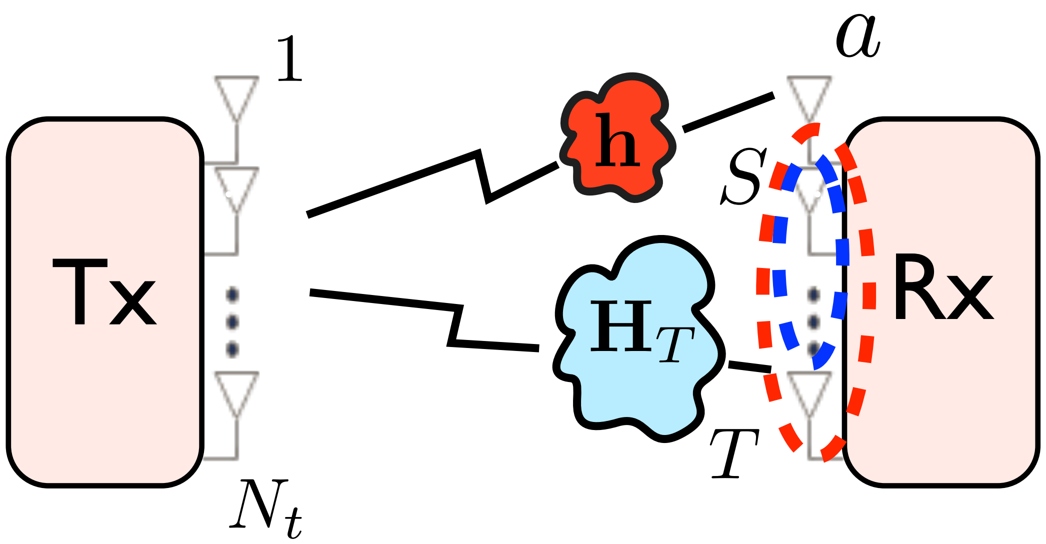

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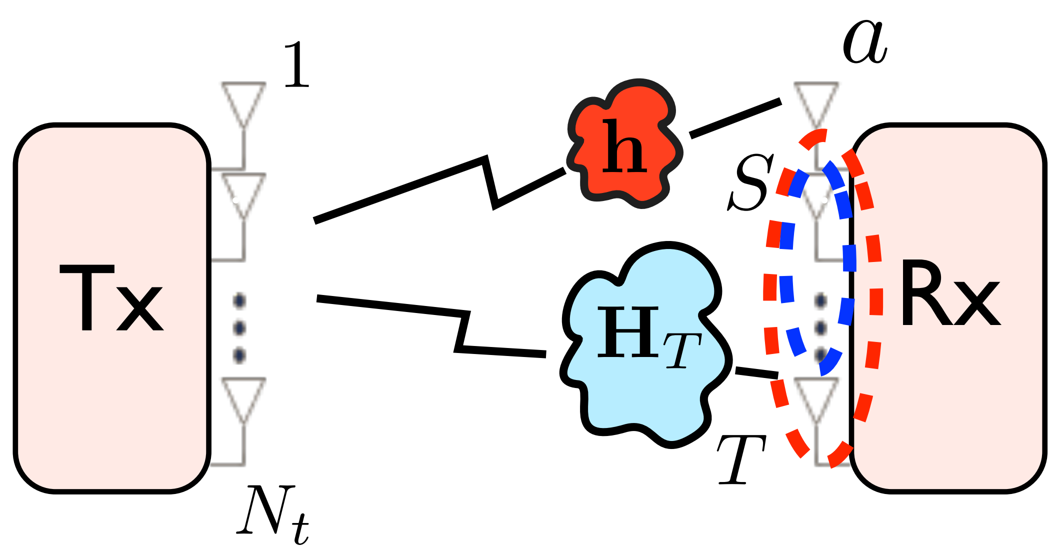

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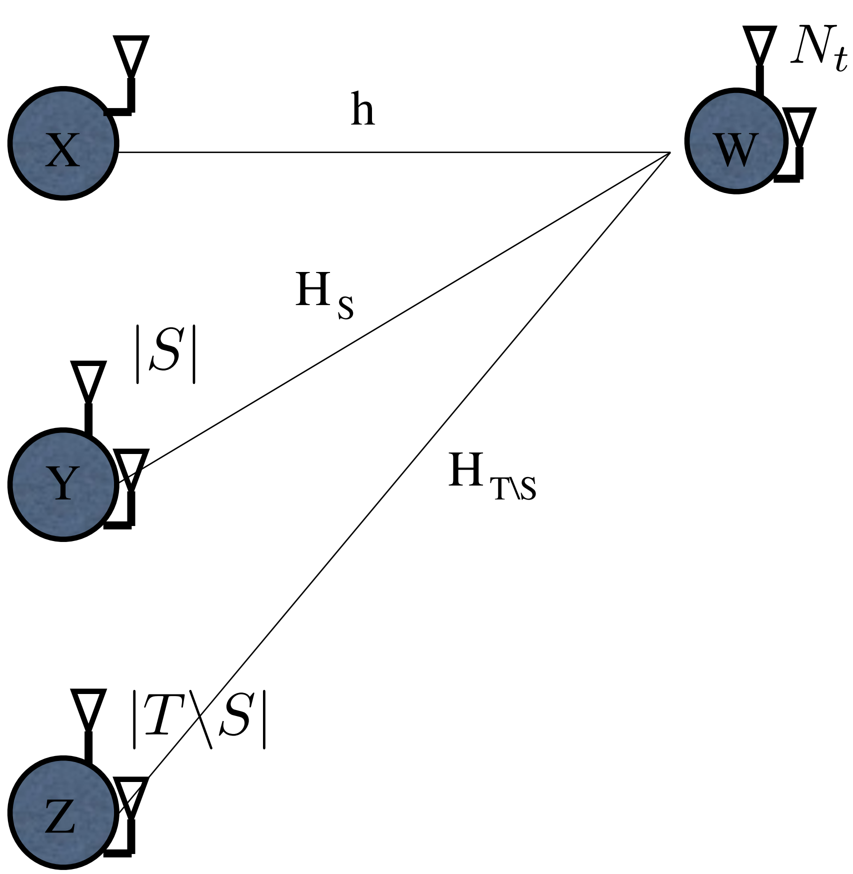

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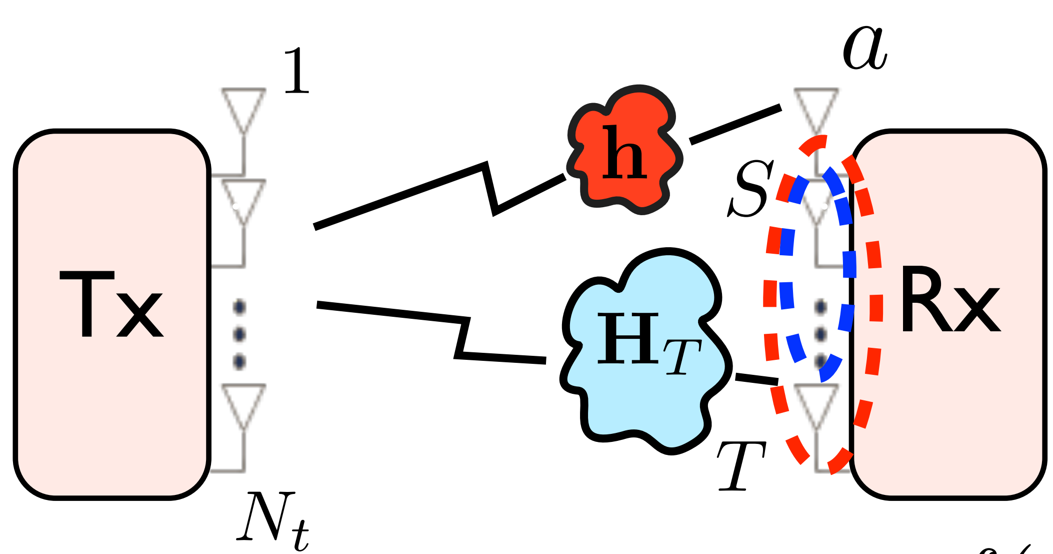

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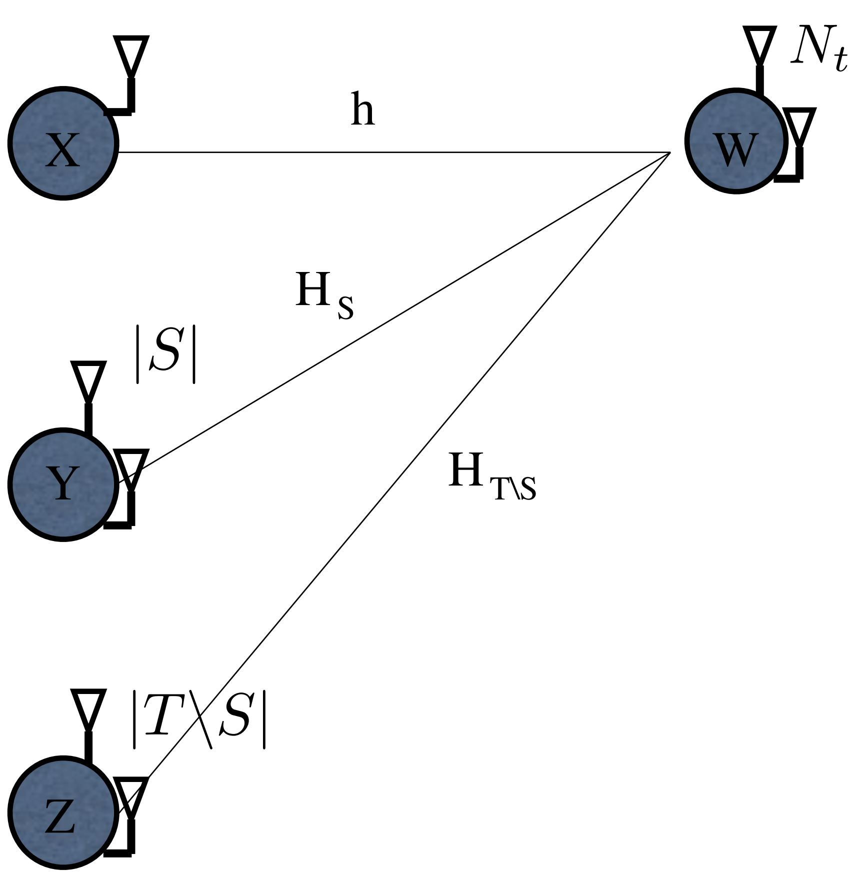

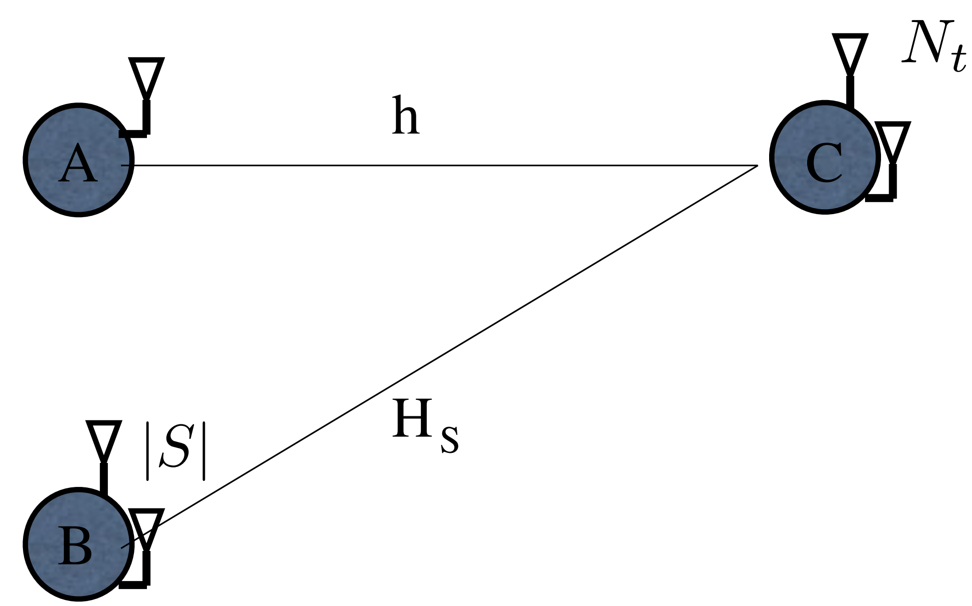

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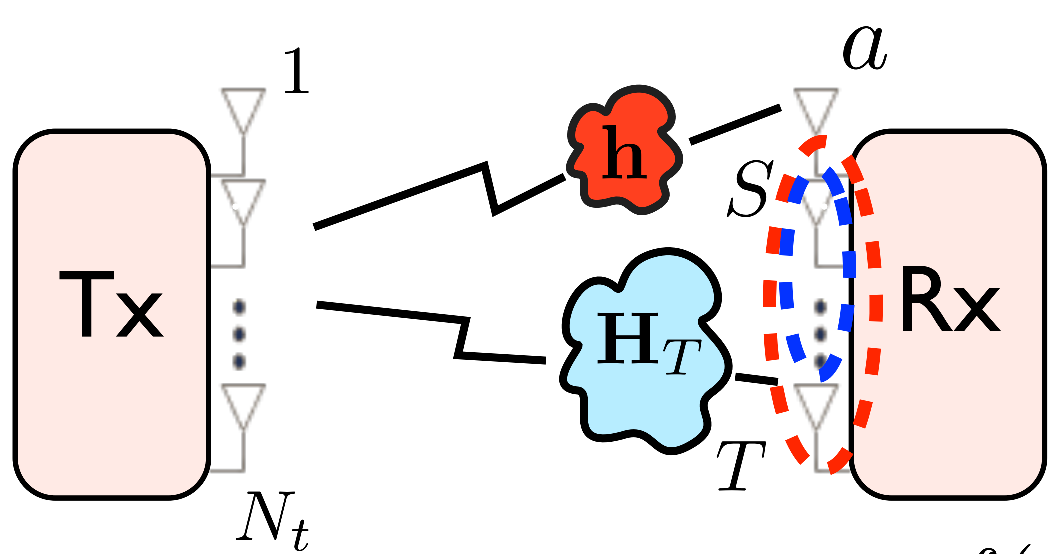

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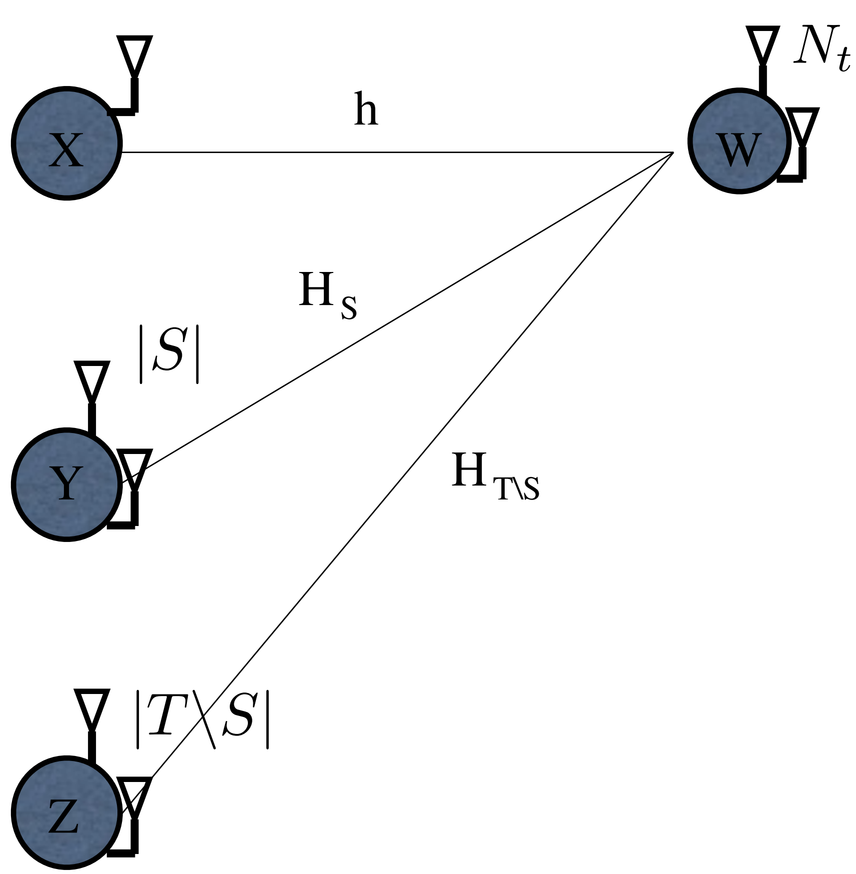
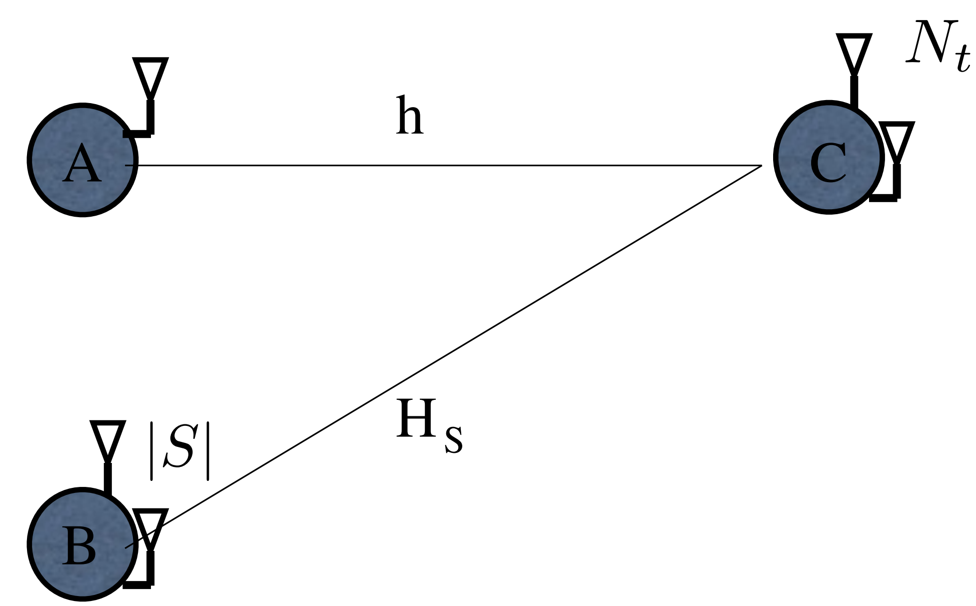


Physically Degraded



Result

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$$\text{Start with } \mathcal{R}_L = \Phi$$

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At step $i, \mathcal{R}_L = \mathcal{R}_L \cup \{i^\star\},$ Choose the best among the rest

$$i^\star = \arg \max_{i \in \{1,2,\ldots,N_r\}, i \notin \mathcal{R}_L} \log\det\left(\mathbf{I} + \frac{P}{N_t}\mathbf{H}_{\mathcal{R}_L \cup \{i\}}\mathbf{H}^\dagger_{\mathcal{R}_L \cup \{i\}}\right),$$

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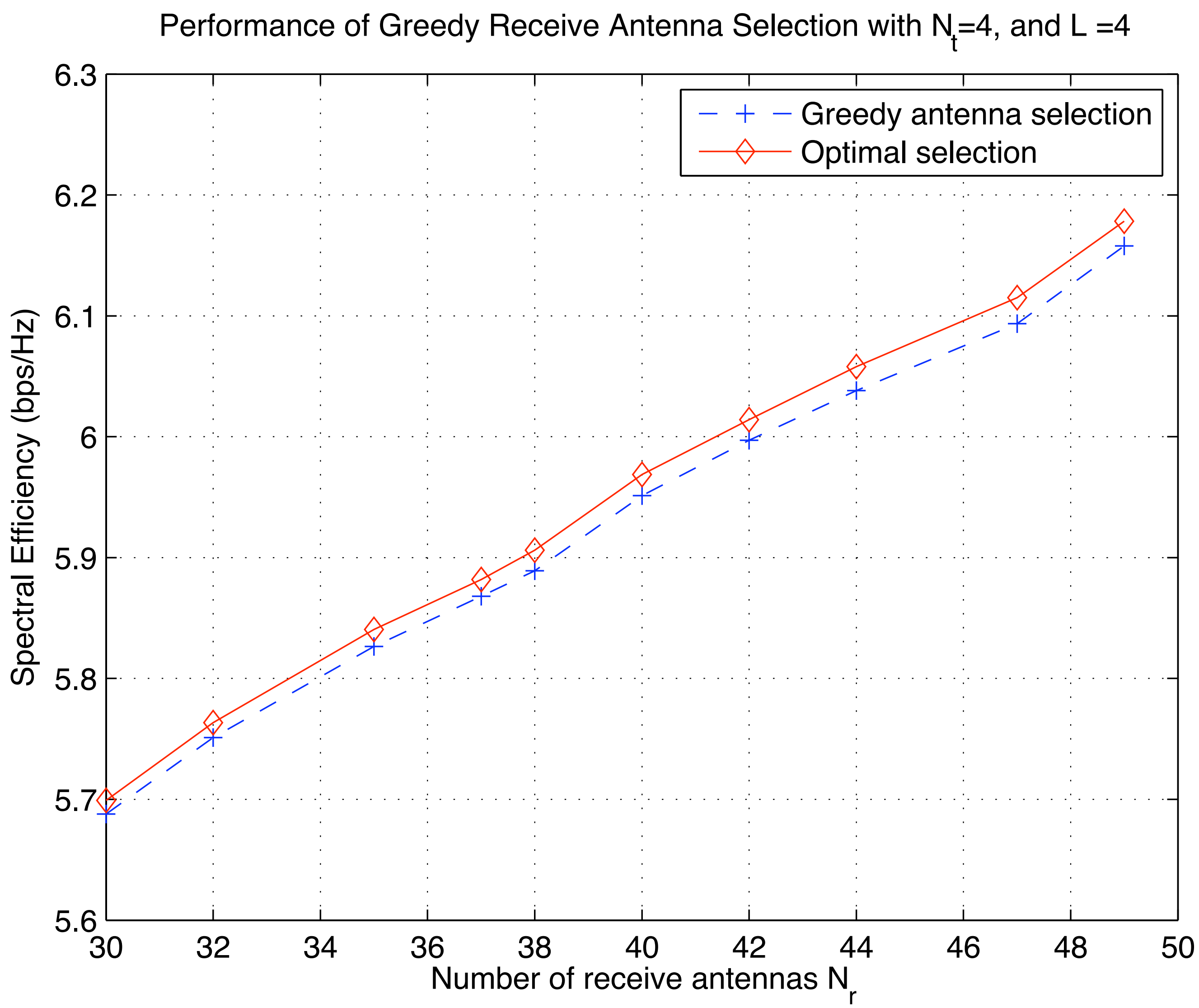
Theorem: *Greedy Algorithm for receive antenna selection with linear complexity achieves at least* $(1 - 1/e)$ *fraction of the optimal solution.*

Simulation Result



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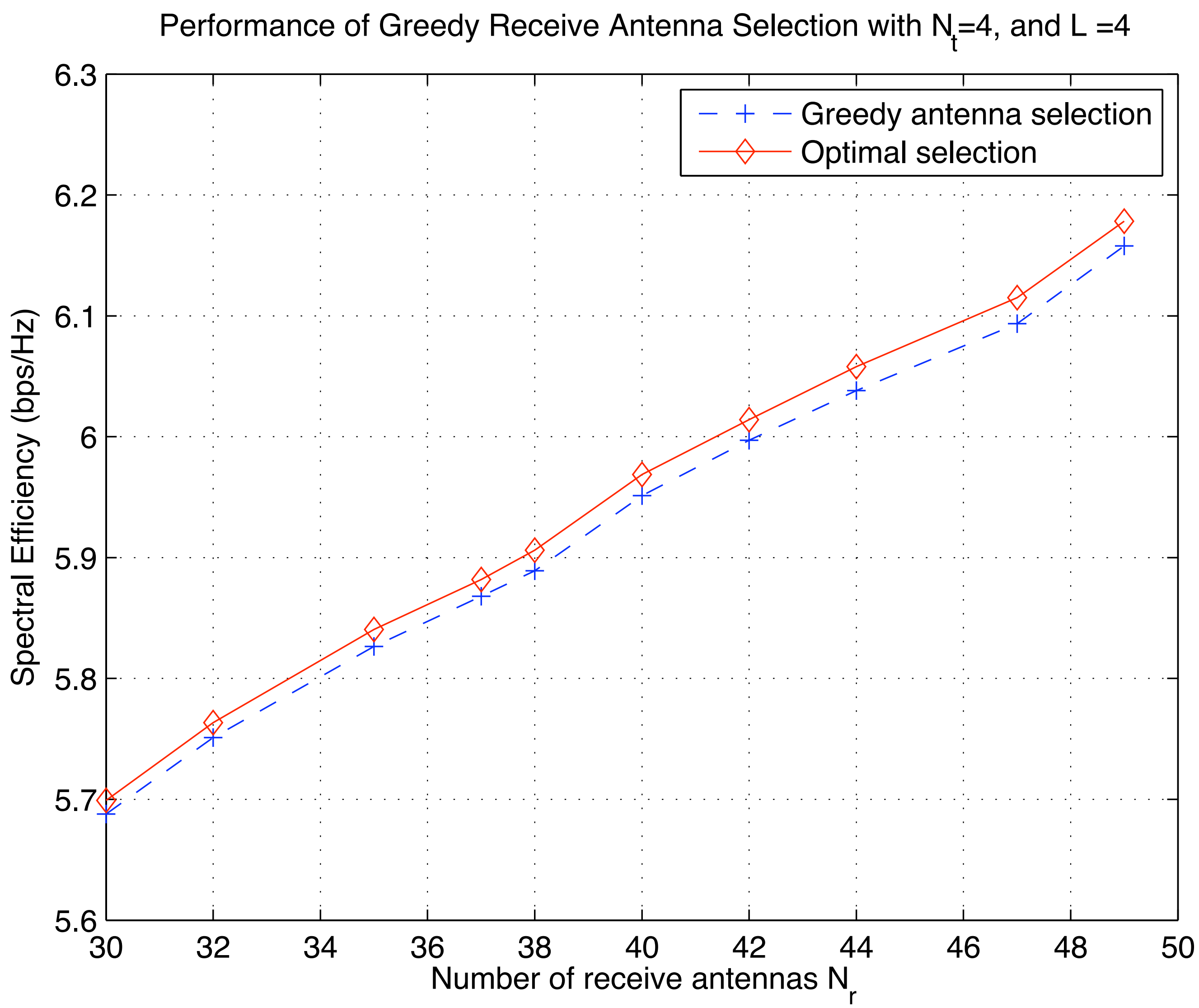
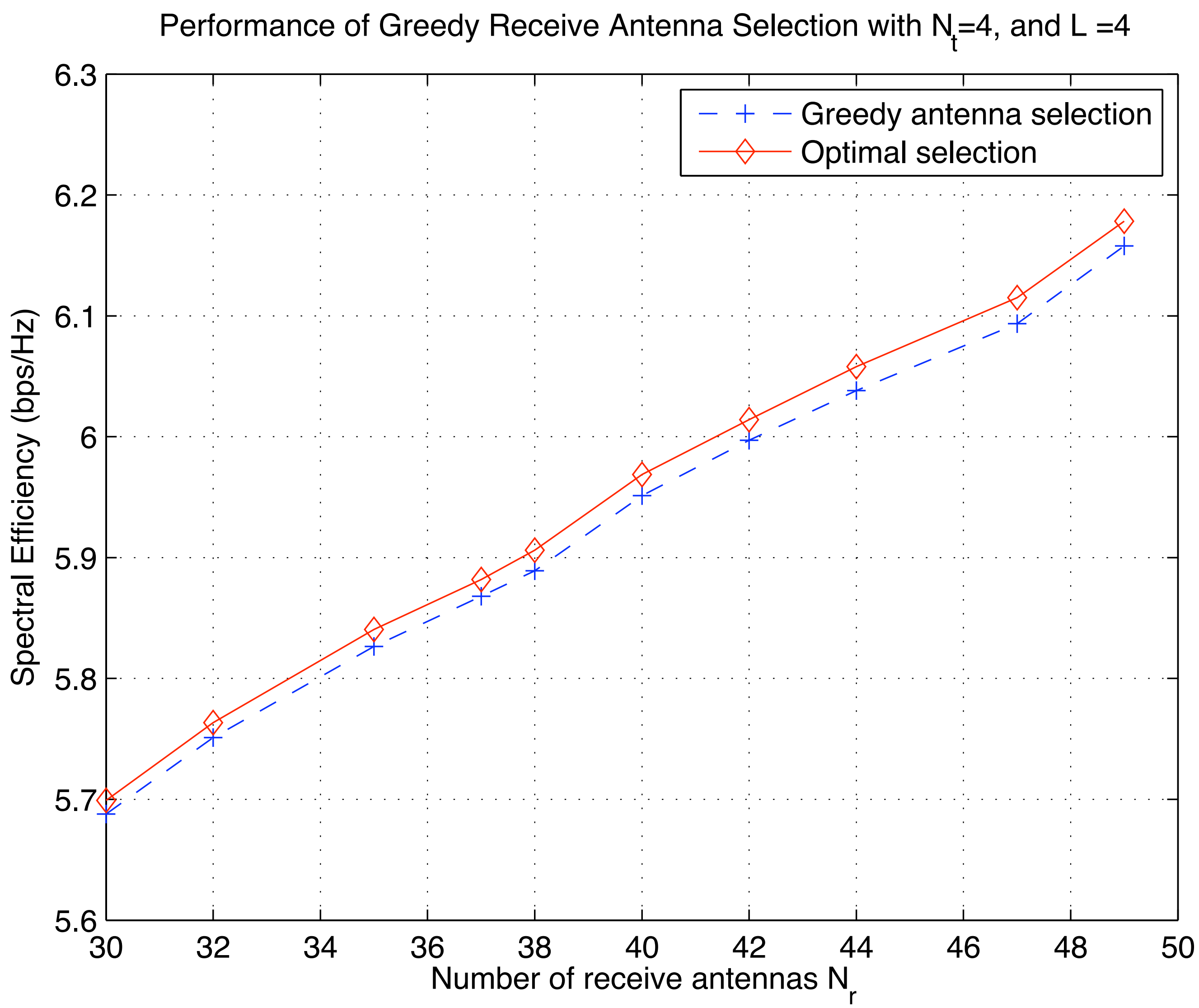


Massive MIMO

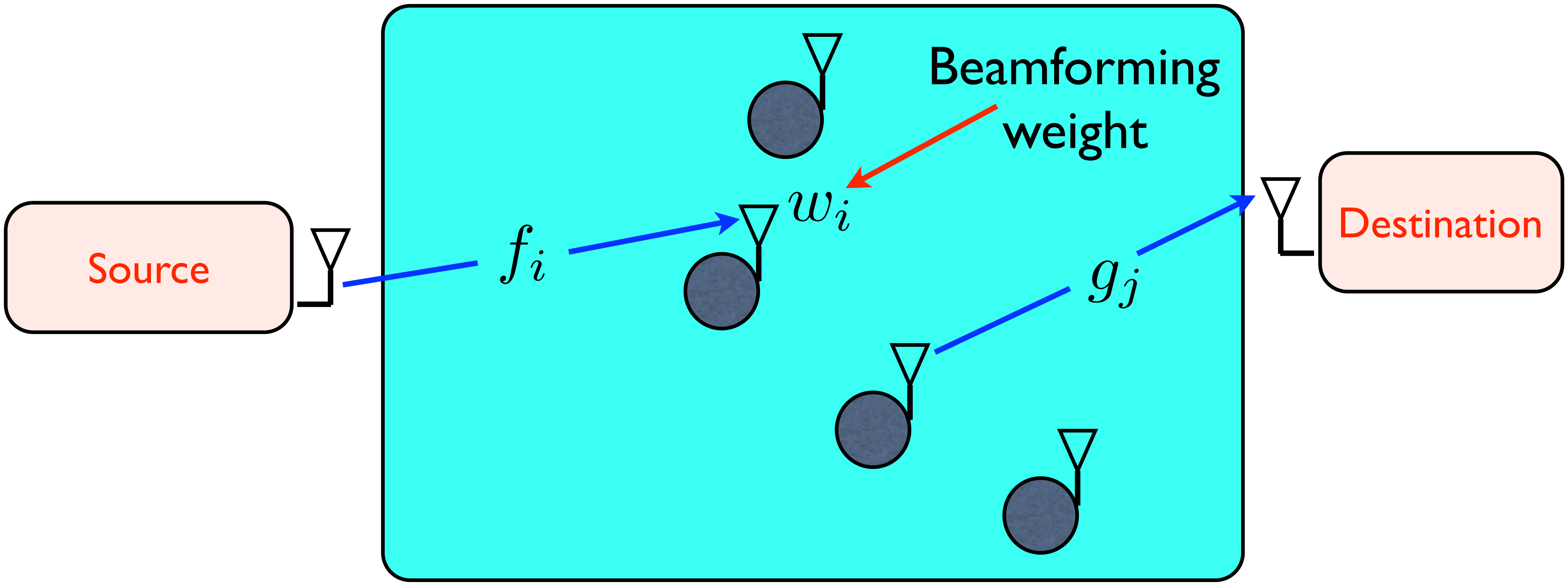
Simulation Result



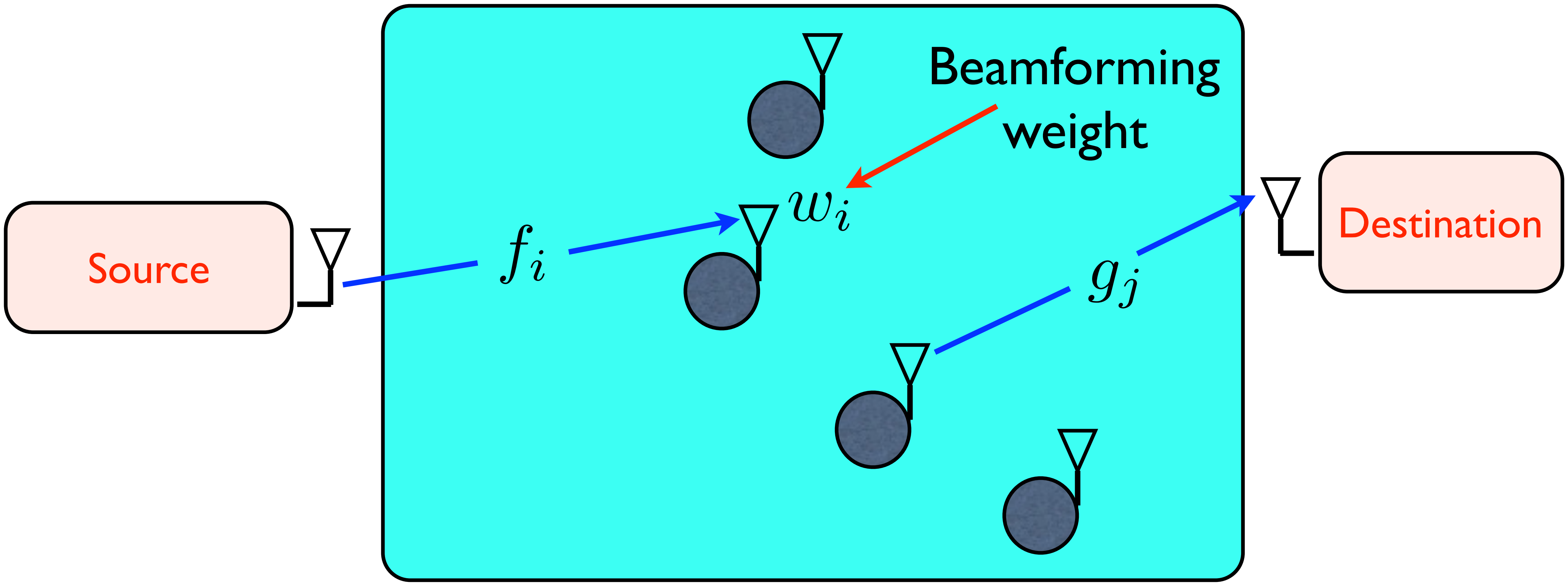
Massive MIMO

Performance Better than promised

Relay Selection



Relay Selection



Find the *L* relay antennas subset that maximizes the mutual information

$$\max_{\mathcal{T}_L \subseteq \{1,2,\ldots,N\}} \max_{\mathbf{w}} \log\left(1 + \frac{\mathbf{w}^\dagger \Delta\Delta^\dagger \mathbf{w}}{\mathbf{w}^\dagger(\Sigma\Sigma^\dagger + \mathbf{I})\mathbf{w}}\right)$$

Relay Antenna Selection is Modular

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$$\Delta = \left[\frac{g_{t_1} f_{t_1}}{\gamma_{t_1}}, \ldots, \frac{g_{t_L} f_{t_L}}{\gamma_{t_L}}\right]^T,$$

$$\mathbf{w} = [w_{t_1}, \ldots, w_{t_L}]^T,$$

$$\Sigma = \begin{bmatrix} \frac{g_{t_1}}{\gamma_{t_1}} & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \frac{g_{t_L}}{\gamma_{t_L}} \end{bmatrix}$$

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$$\max_{\mathbf{x}} \frac{\mathbf{x}^T \mathbf{A} \mathbf{x}}{\mathbf{x}^T \mathbf{B} \mathbf{x}}$$

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Rayleigh-Ritz Theorem

$$\max_{\mathcal{T}_L \subseteq \{1,2,\ldots,N\}} \sum_{i \in \mathcal{T}_L} \frac{|g_i|^2 |f_i|^2}{|f_i|^2 + |g_i|^2 + 1}$$

Relay Antenna Selection is Modular

$$\max_{\mathcal{T}_L \subseteq \{1,2,\ldots,N\}} \max_{\mathbf{w}} \log\left(1 + \frac{\mathbf{w}^\dagger \Delta \Delta^\dagger \mathbf{w}}{\mathbf{w}^\dagger(\Sigma\Sigma^\dagger + \mathbf{I})\mathbf{w}}\right)$$

$$\Delta = \left[\frac{g_{t_1} f_{t_1}}{\gamma_{t_1}}, \ldots, \frac{g_{t_L} f_{t_L}}{\gamma_{t_L}}\right]^T,$$

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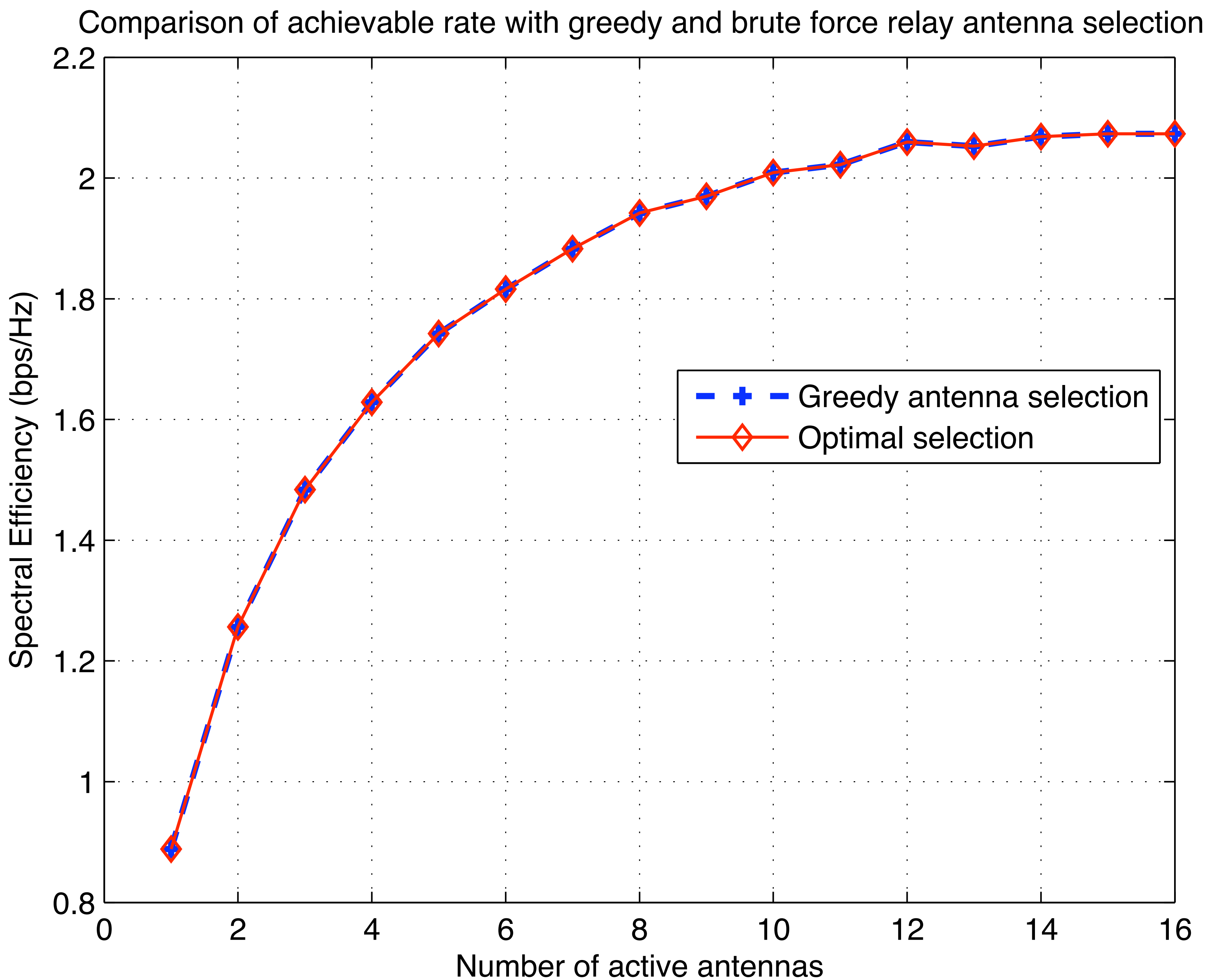
Relay Selection Problem is Modular

Result

Result

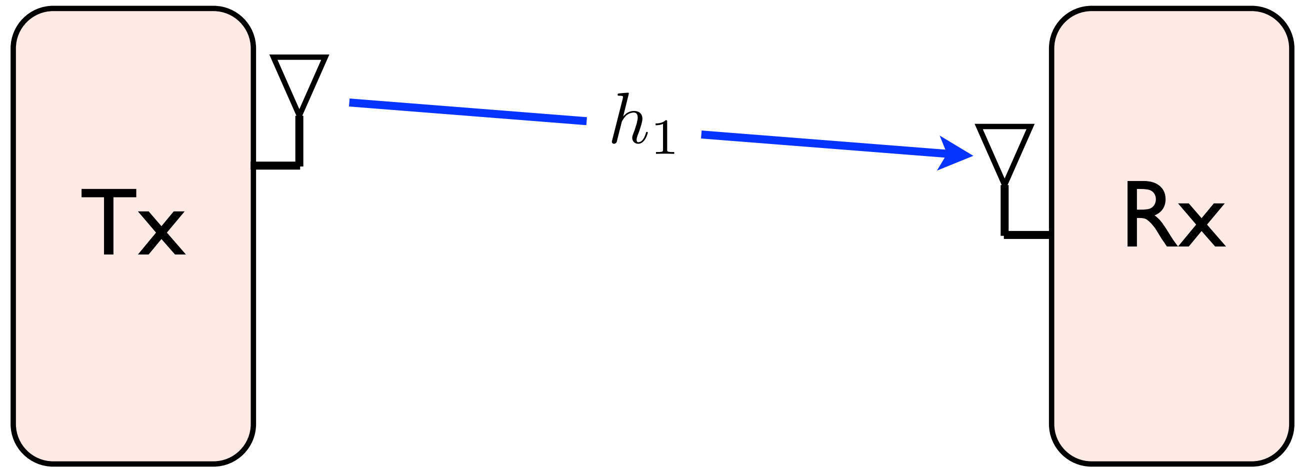
Theorem: *Greedy Algorithm for relay antenna selection with linear complexity achieves the optimal solution.*

Simulation Result

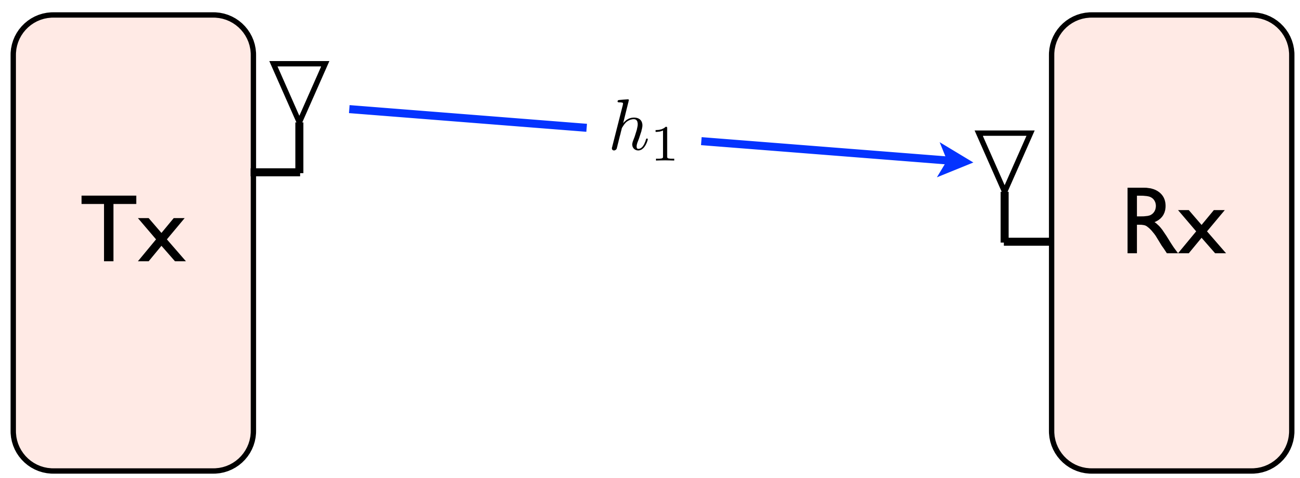


Some Counter-Examples

Transmit Antenna Selection

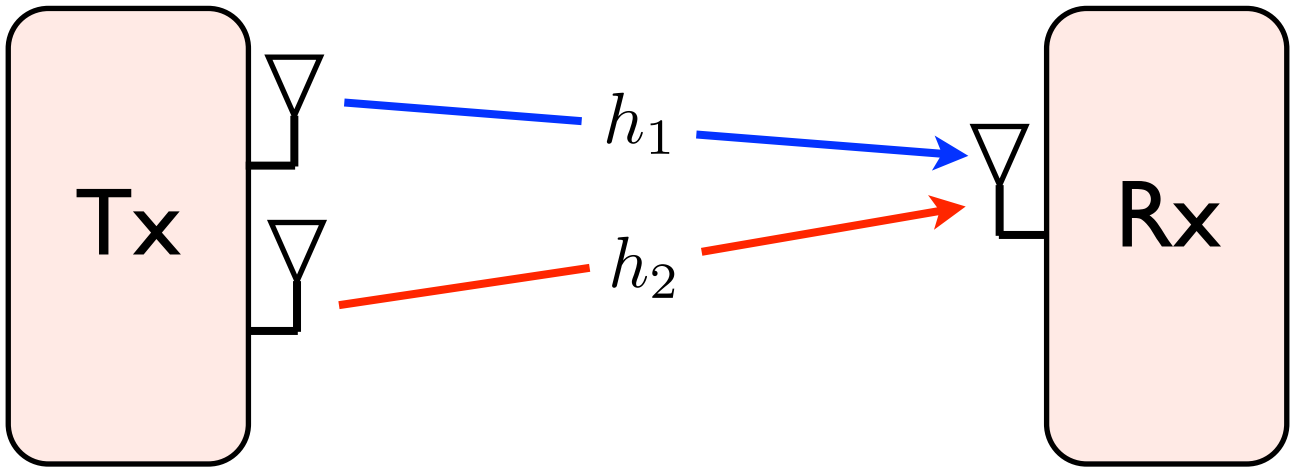


Transmit Antenna Selection



$$C_{\{1\}} = \log(1 + P|h_1|^2)$$

Transmit Antenna Selection



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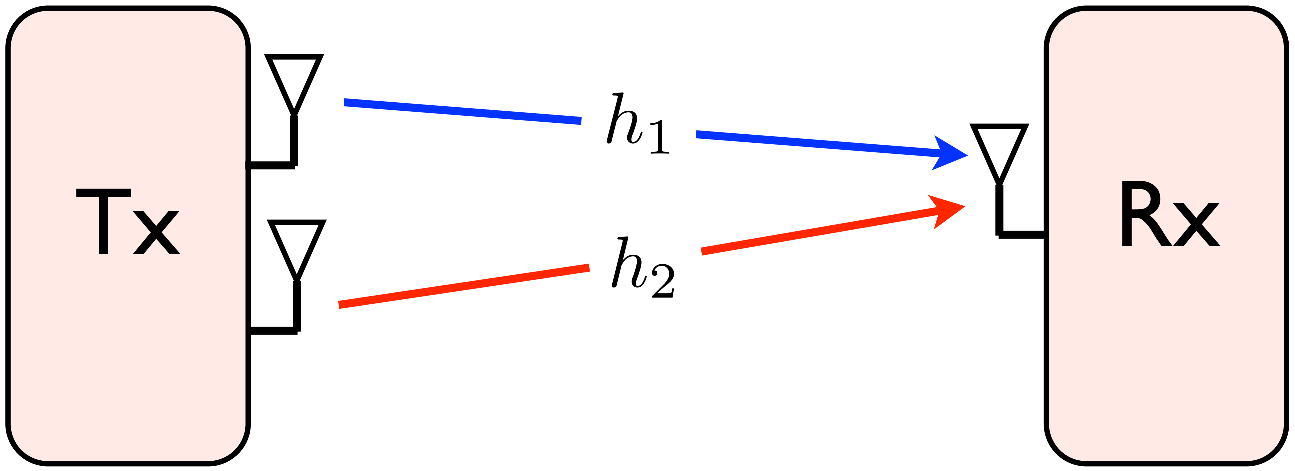
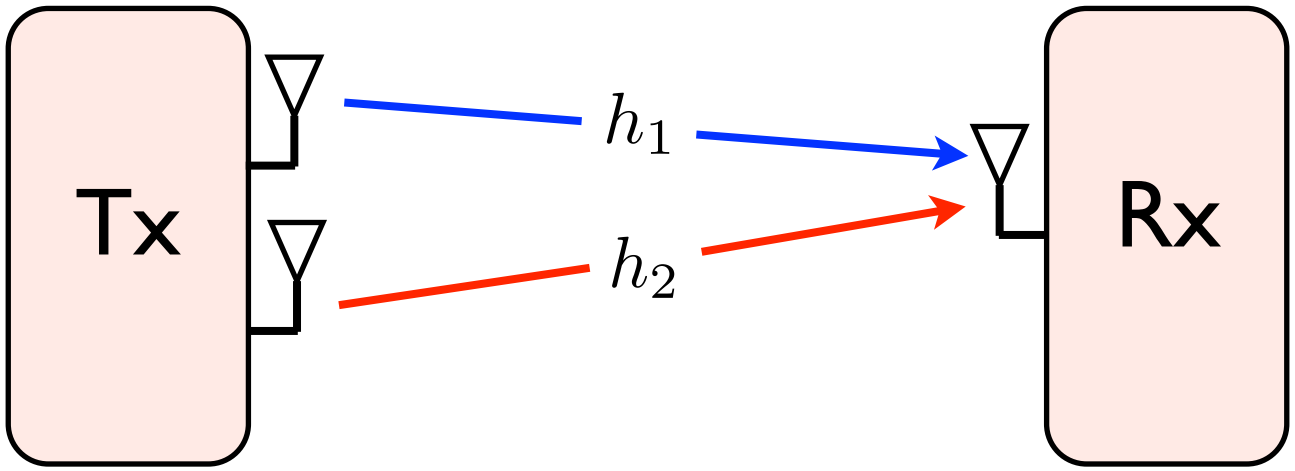
Transmit Antenna Selection



$$C_{\{1\}} = \log(1 + P|h_1|^2) \qquad\qquad C_{\{1,2\}} = \log\left(1 + \frac{P}{2}(|h_1|^2 + |h_2|^2)\right)$$

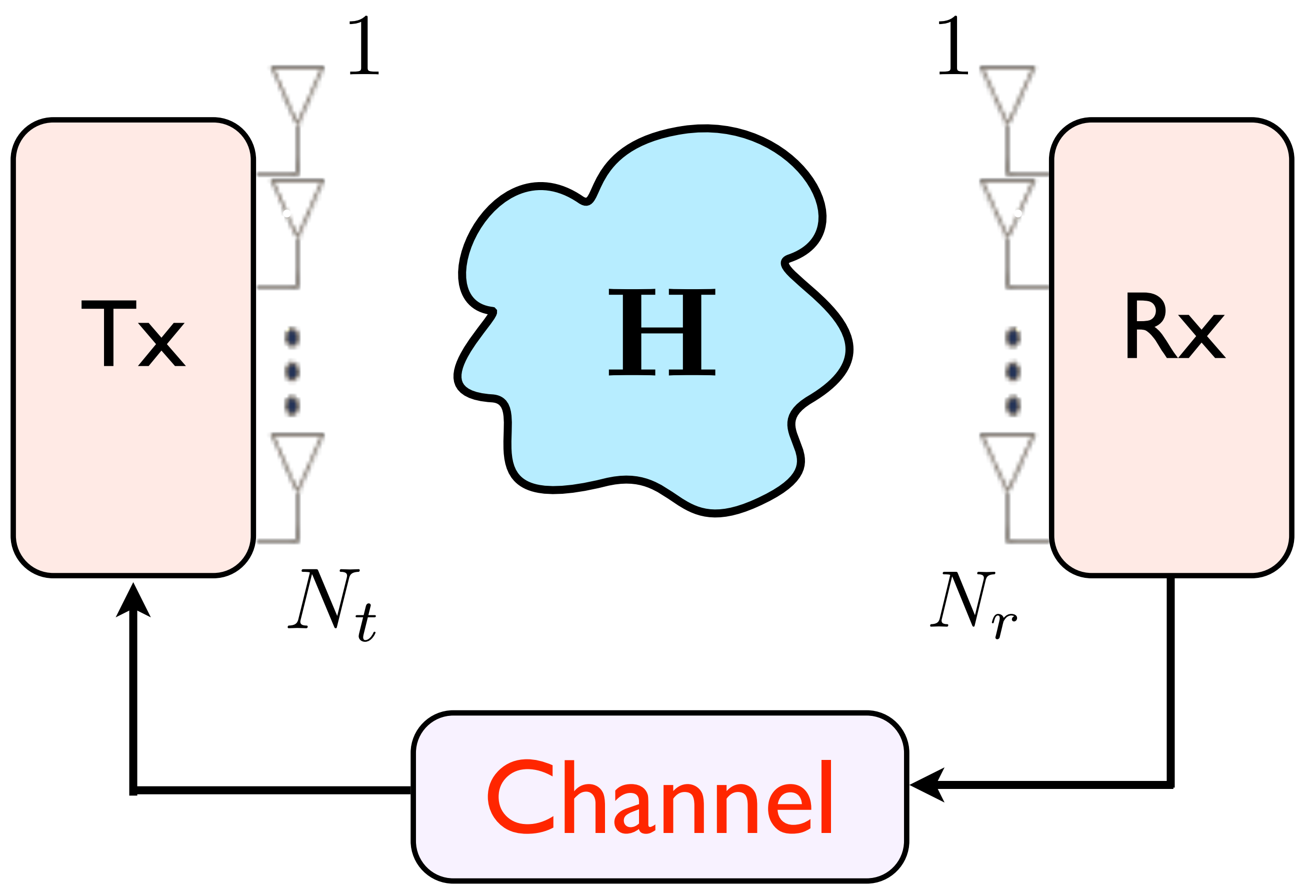
Transmit Antenna Selection



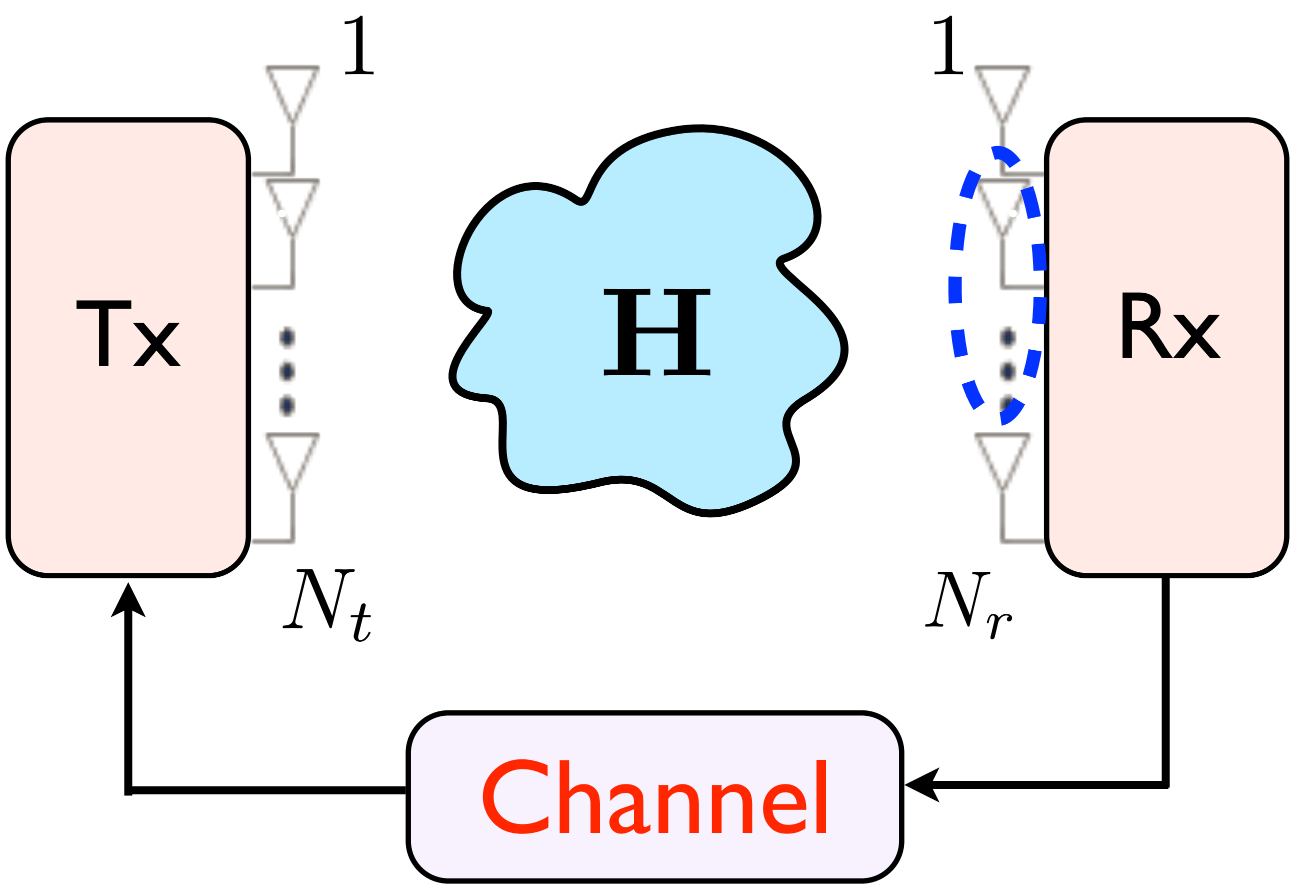
$$C_{\{1\}} = \log(1 + P|h_1|^2) \qquad C_{\{1,2\}} = \log\left(1 + \frac{P}{2}(|h_1|^2 + |h_2|^2)\right)$$

Transmit Antenna Selection is NOT Monotone

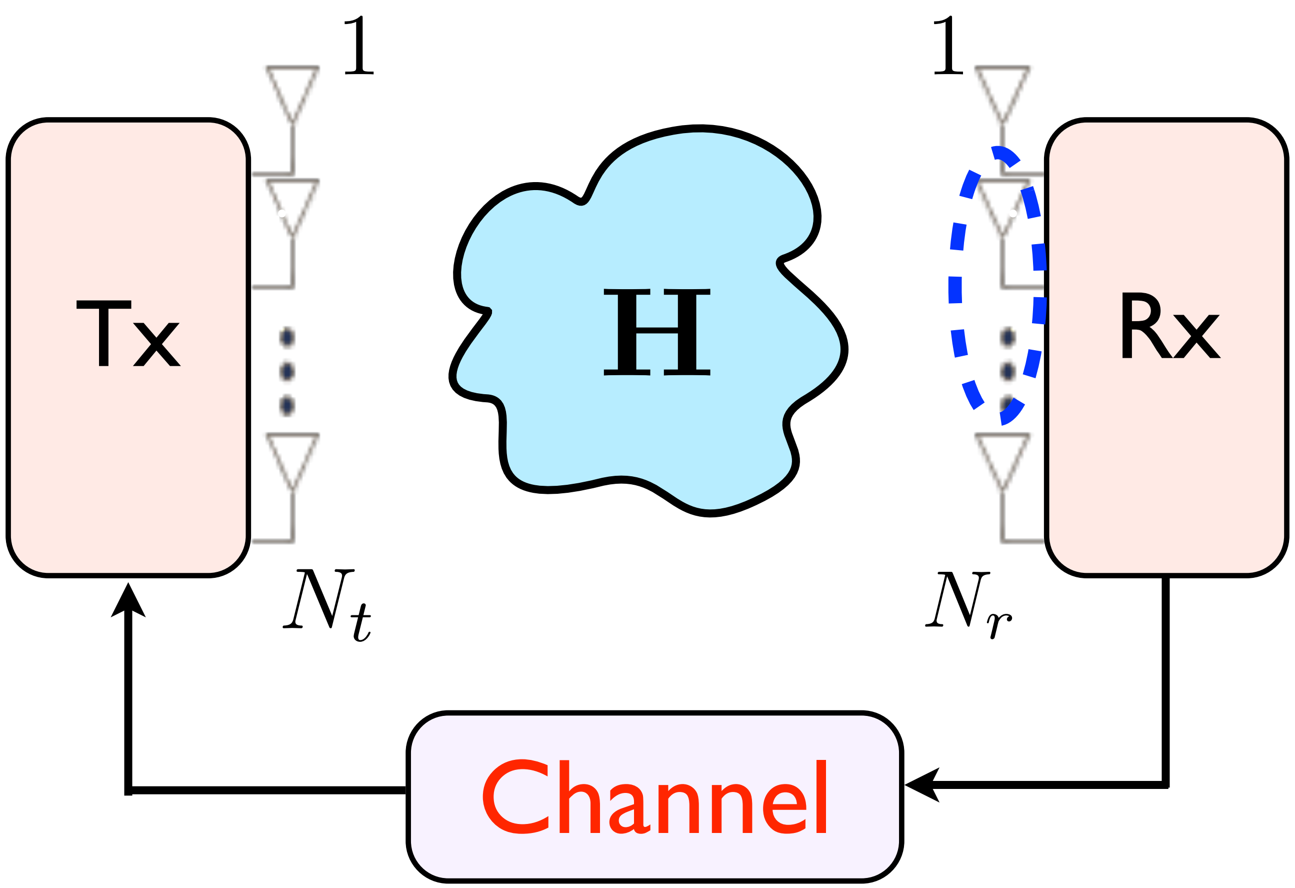
Receive Antenna Selection with CSIT



Receive Antenna Selection with CSIT

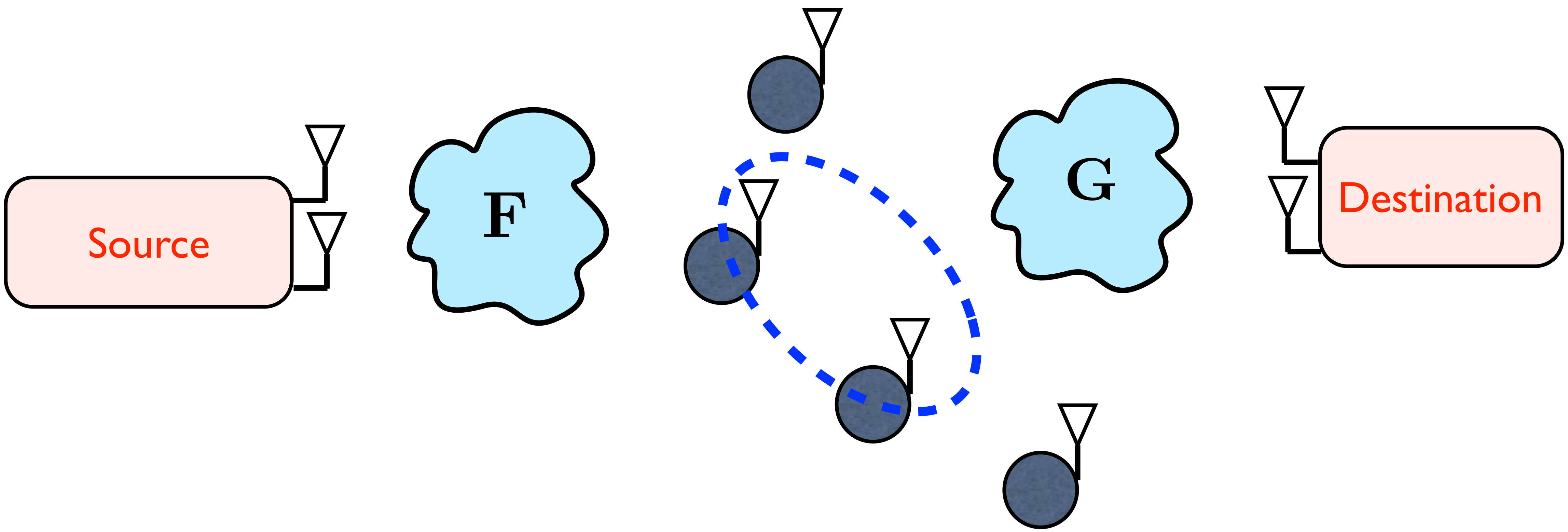


Receive Antenna Selection with CSIT

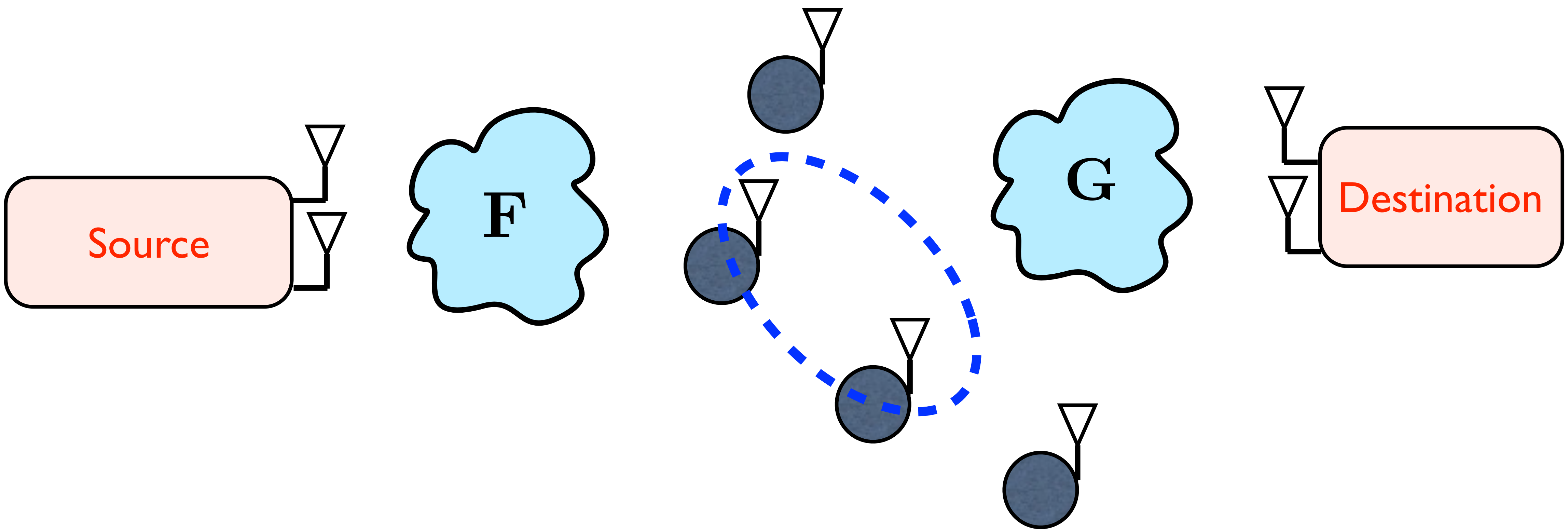


With Watefilling Receive Antenna Selection is Monotone
but NOT Sub-modular

Relay Antenna Selection with MIMO Source-Destination



Relay Antenna Selection with MIMO Source-Destination



With MIMO S-D Relay Antenna Selection is NOT Sub-modular

Conclusions

- Theoretical Bounds on Greedy Algorithms
- Relay case - Greedy is Optimal
- Most other antenna selection problems are not sub-modular