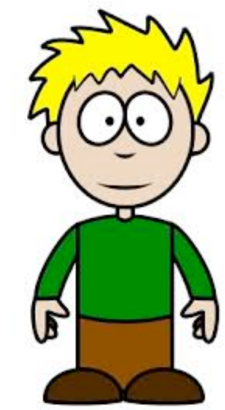
Approximating the FDMA Capacity

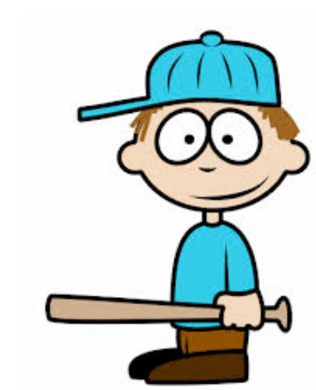
Rahul Vaze

Kiran Koshy

Andrew Thangaraj

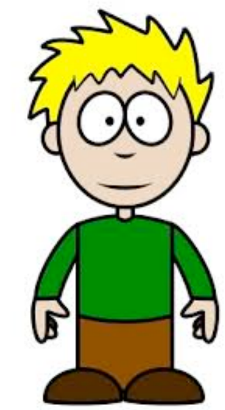
Partitioning Problem

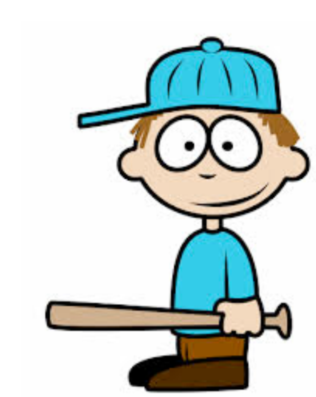




Partitioning Problem

S_i



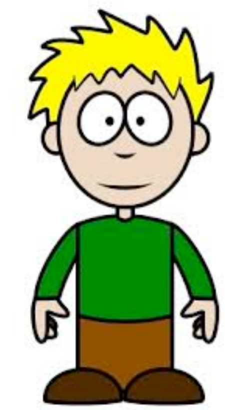


Partitioning Problem

$f_1(S_i)$

S_i

$f_2(S_i)$



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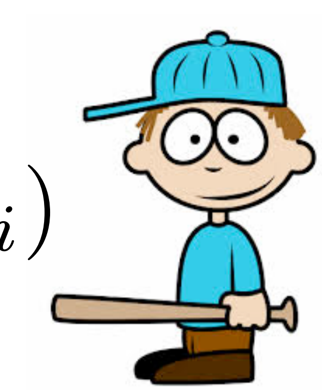

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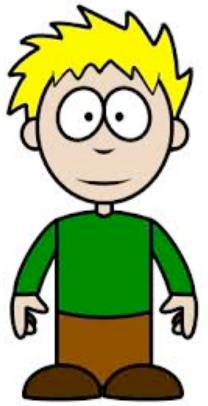

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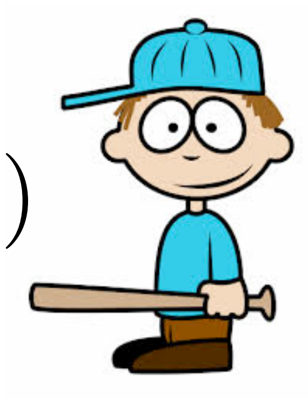

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$$max \sum_{i=1}^{n} f_i(S_i)$$

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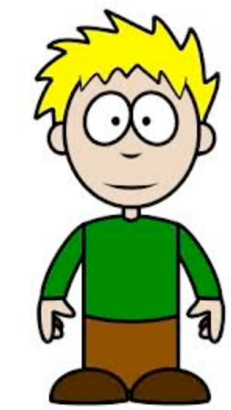

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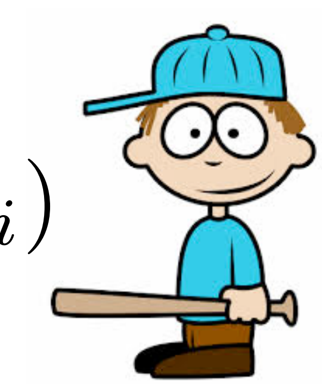

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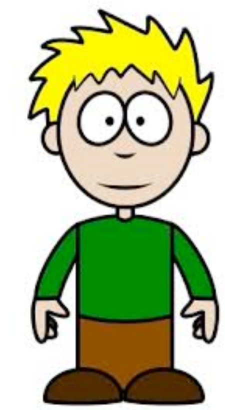

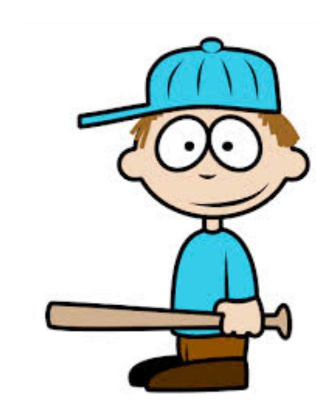

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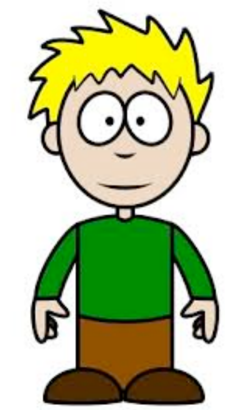

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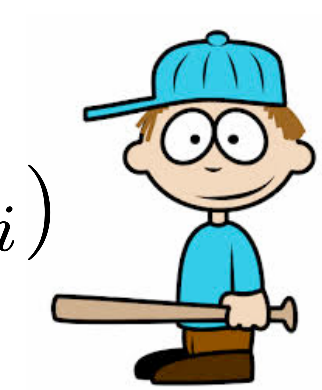

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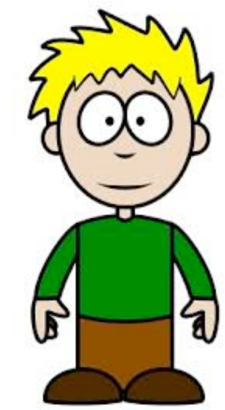

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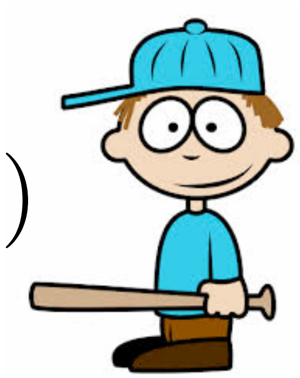

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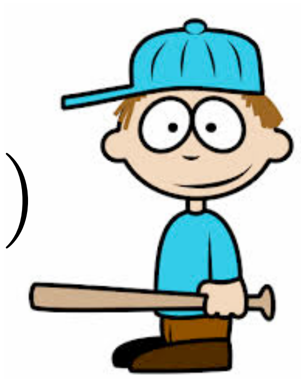

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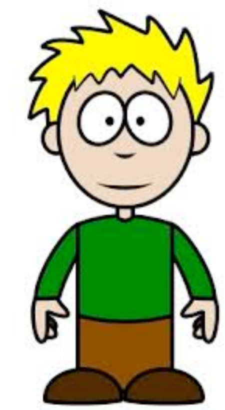

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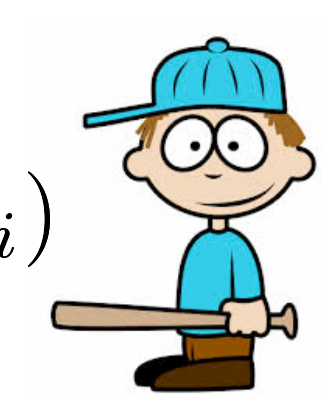

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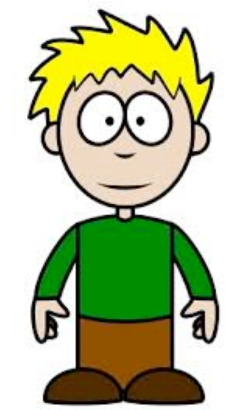

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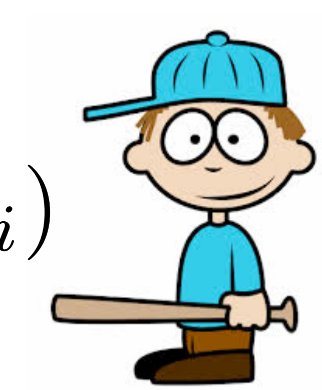

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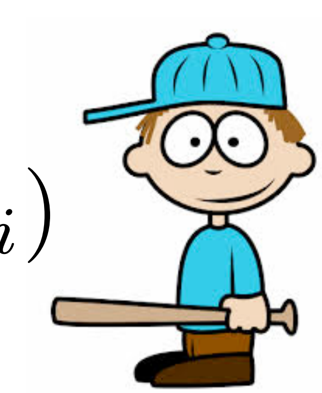

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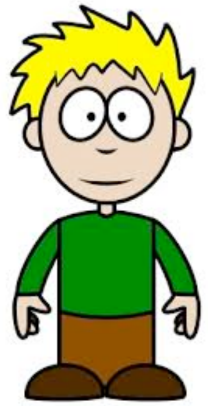

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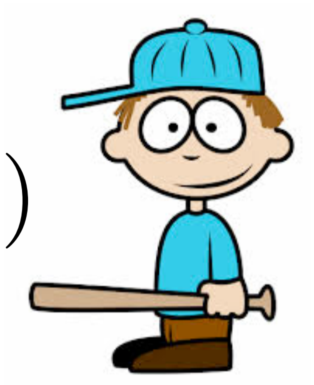

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s_2

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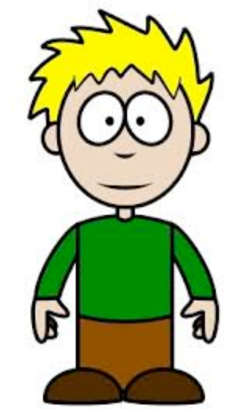

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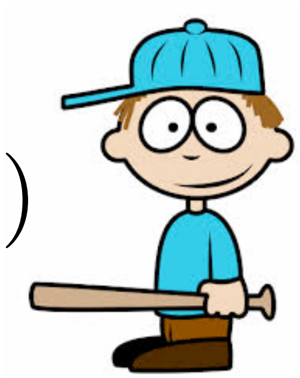

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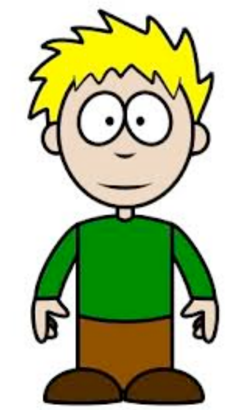

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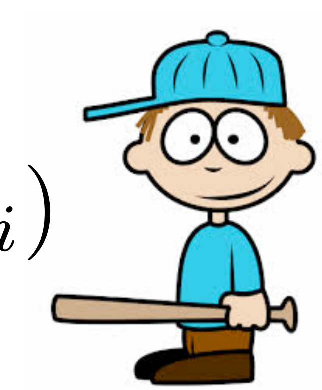

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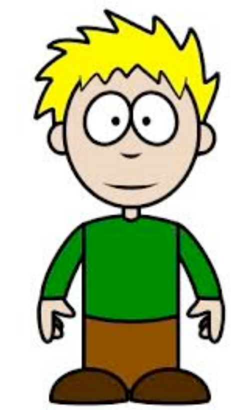

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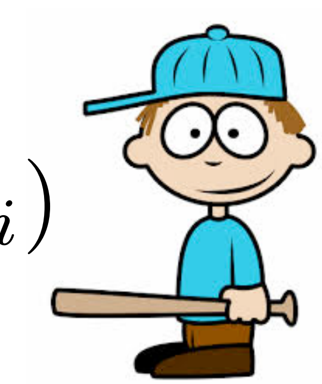

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Greedy Algorithm : *At each step add an element*

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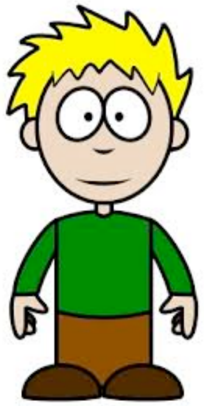

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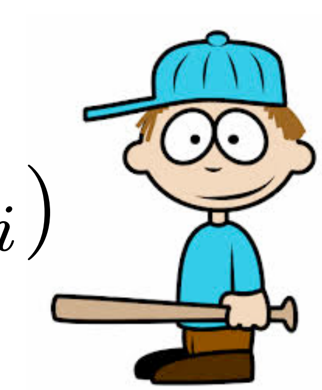
$f_n(S_i)$



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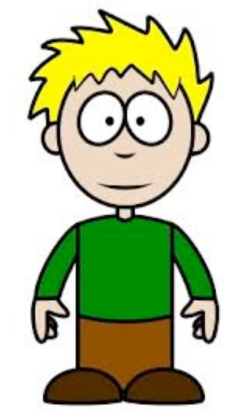

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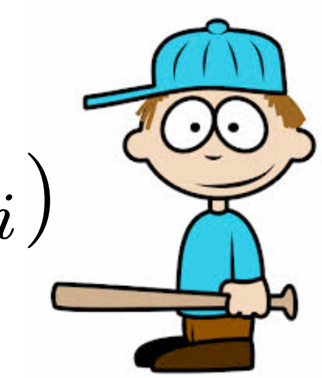

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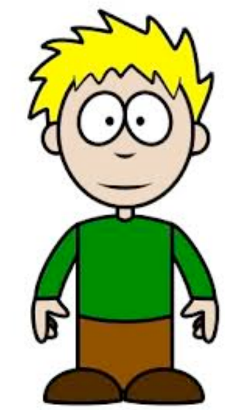

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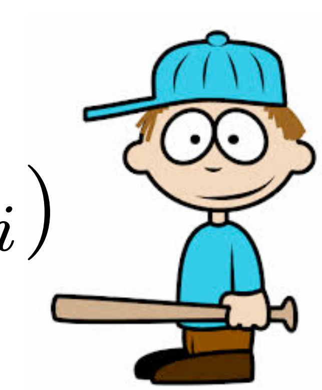

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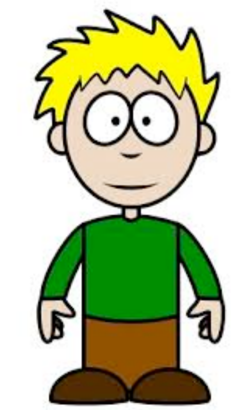

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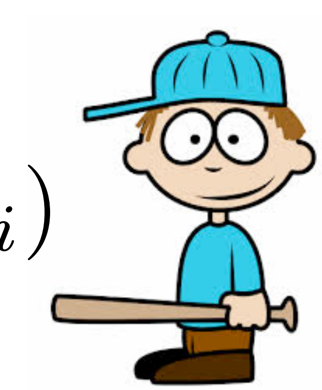

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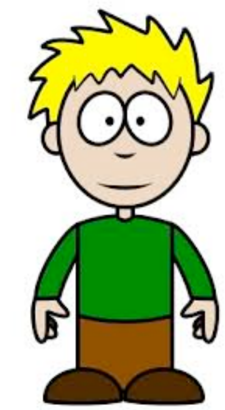

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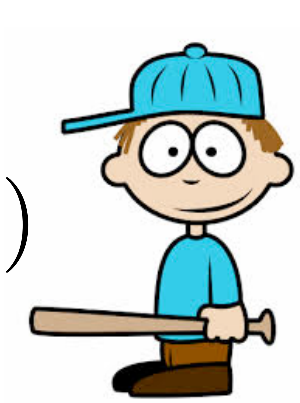

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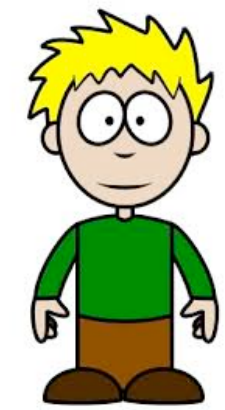

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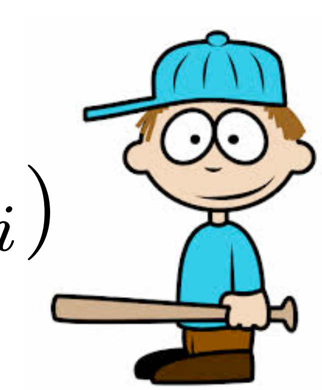

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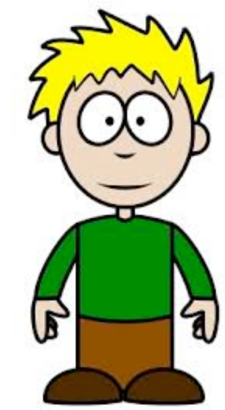

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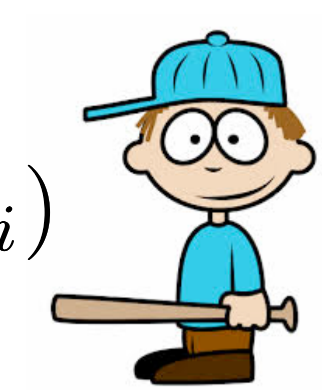

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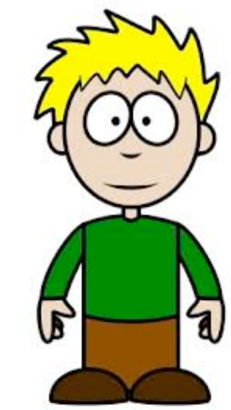

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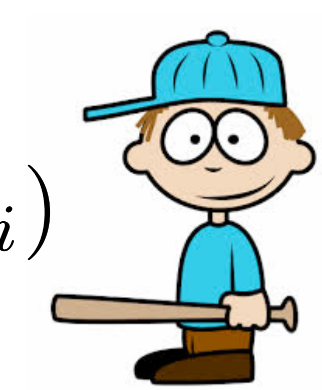

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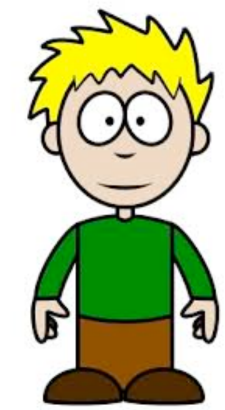

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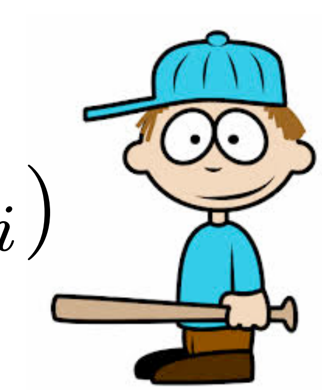

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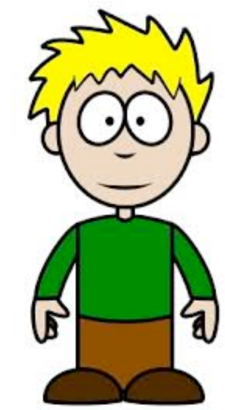

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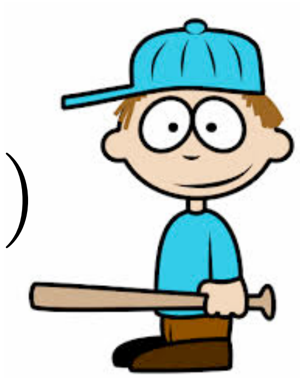

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Diminishing Returns Property: Value of adding an element

to smaller set is more than that of the bigger set

Examples

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$H(x|A)$ $H(x|B)$

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conditioning

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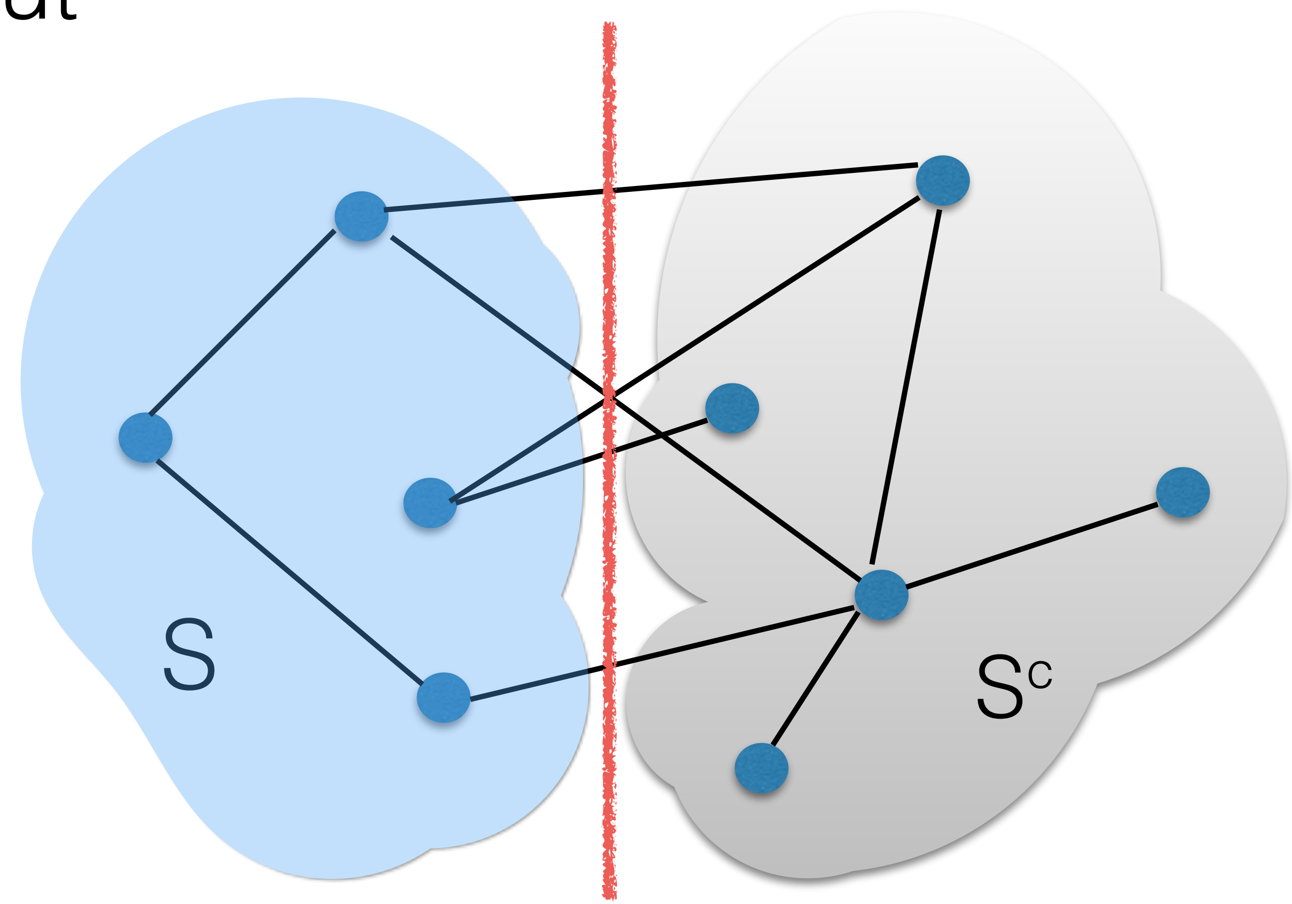
$H(x|A) >= H(x|B)$

conditioning

Mutual Information

Examples

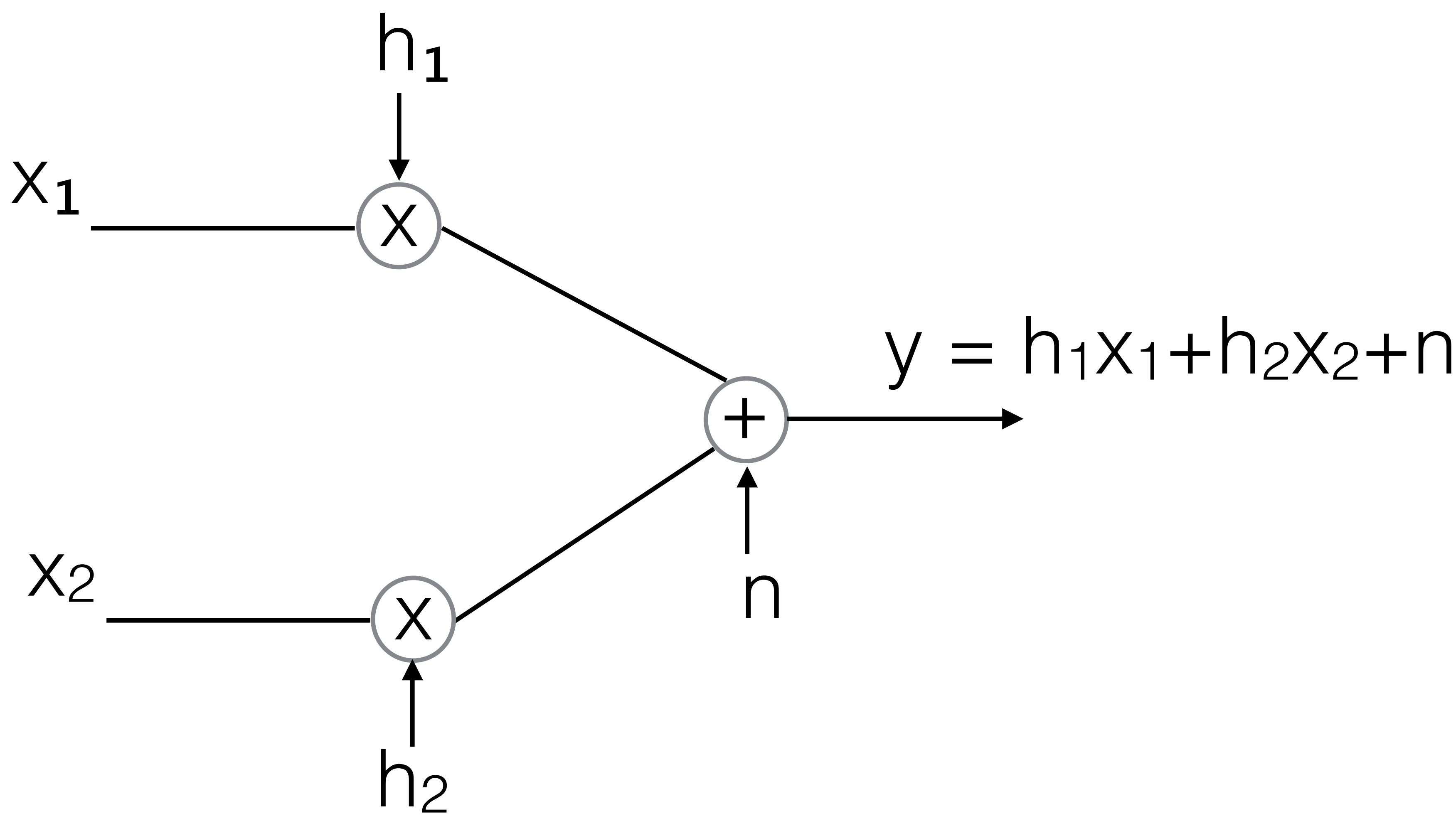
Graph Cut



# of edges crossing the cut (S, S^c)

Multiple Access Channel (MAC)

Two-user



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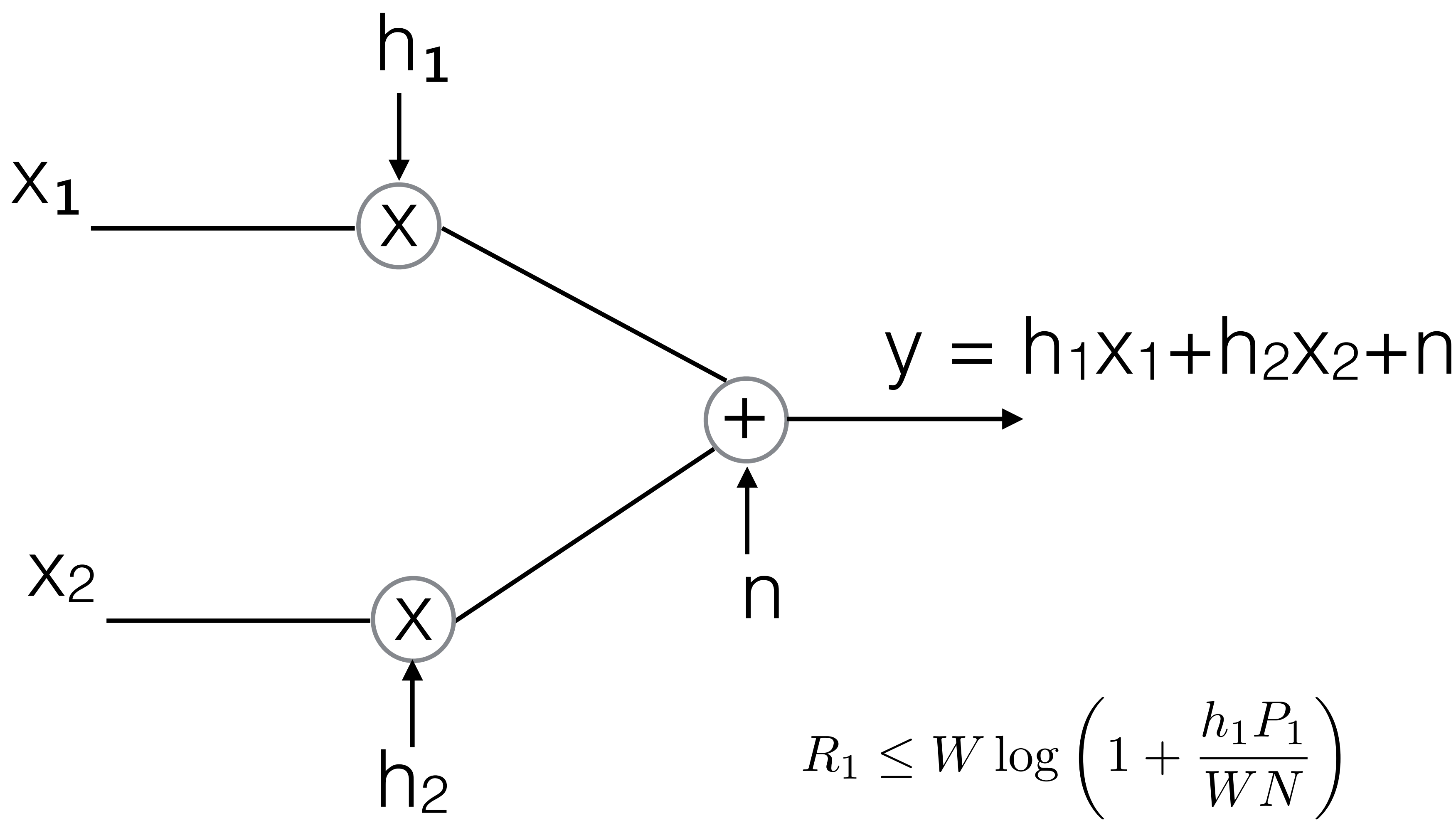

$$R_1 \leq W \log\left(1 + \frac{h_1 P_1}{WN}\right)$$

$$R_1 + R_2 \leq W \log\left(1 + \frac{h_1 P_1 + h_2 P_2}{WN}\right)$$

Multiple Access Channel (MAC)

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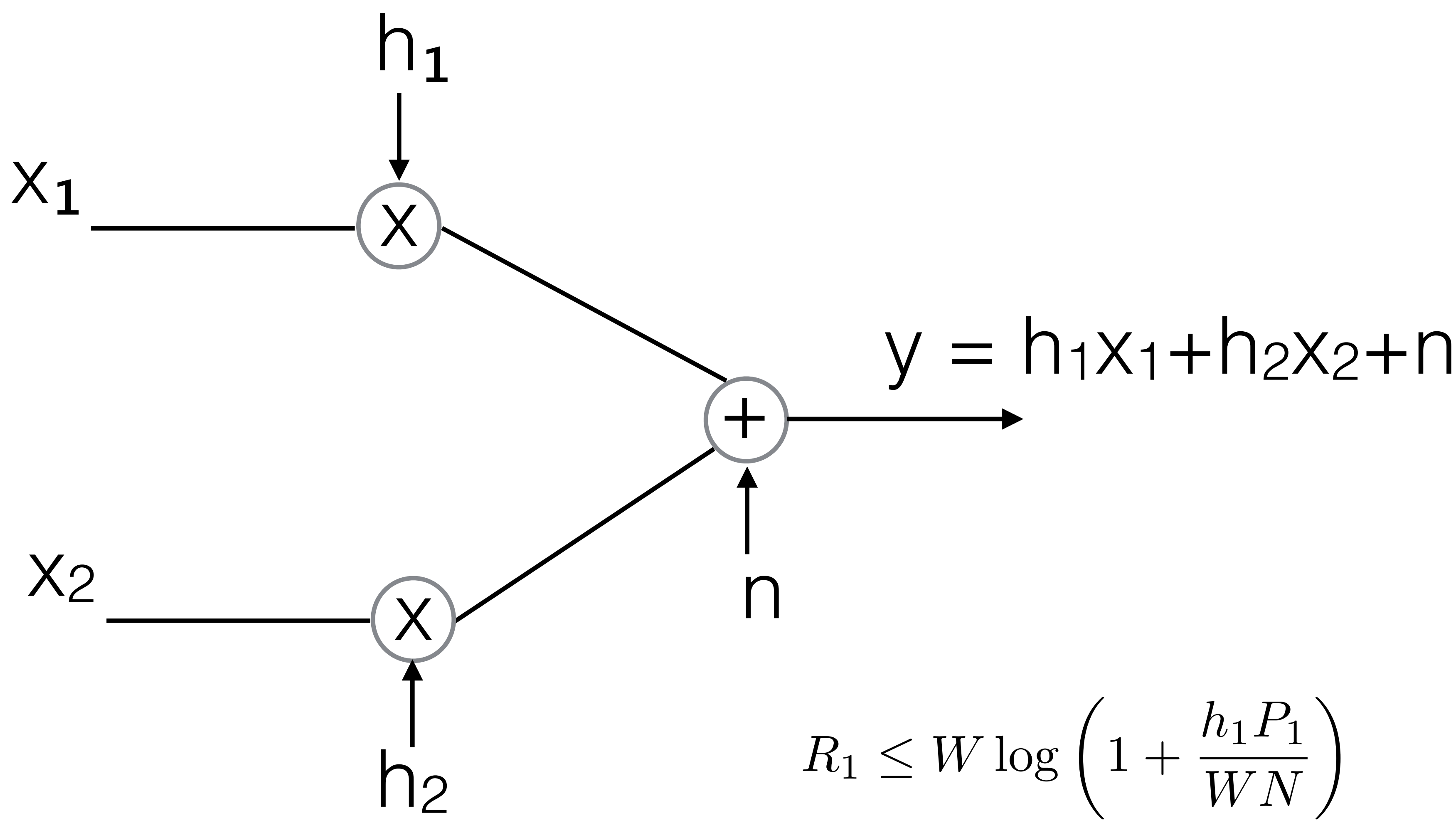

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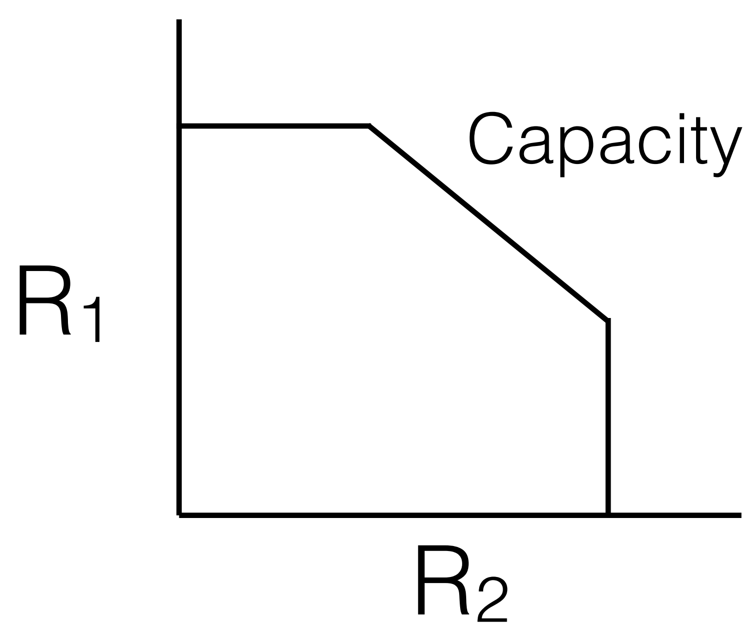

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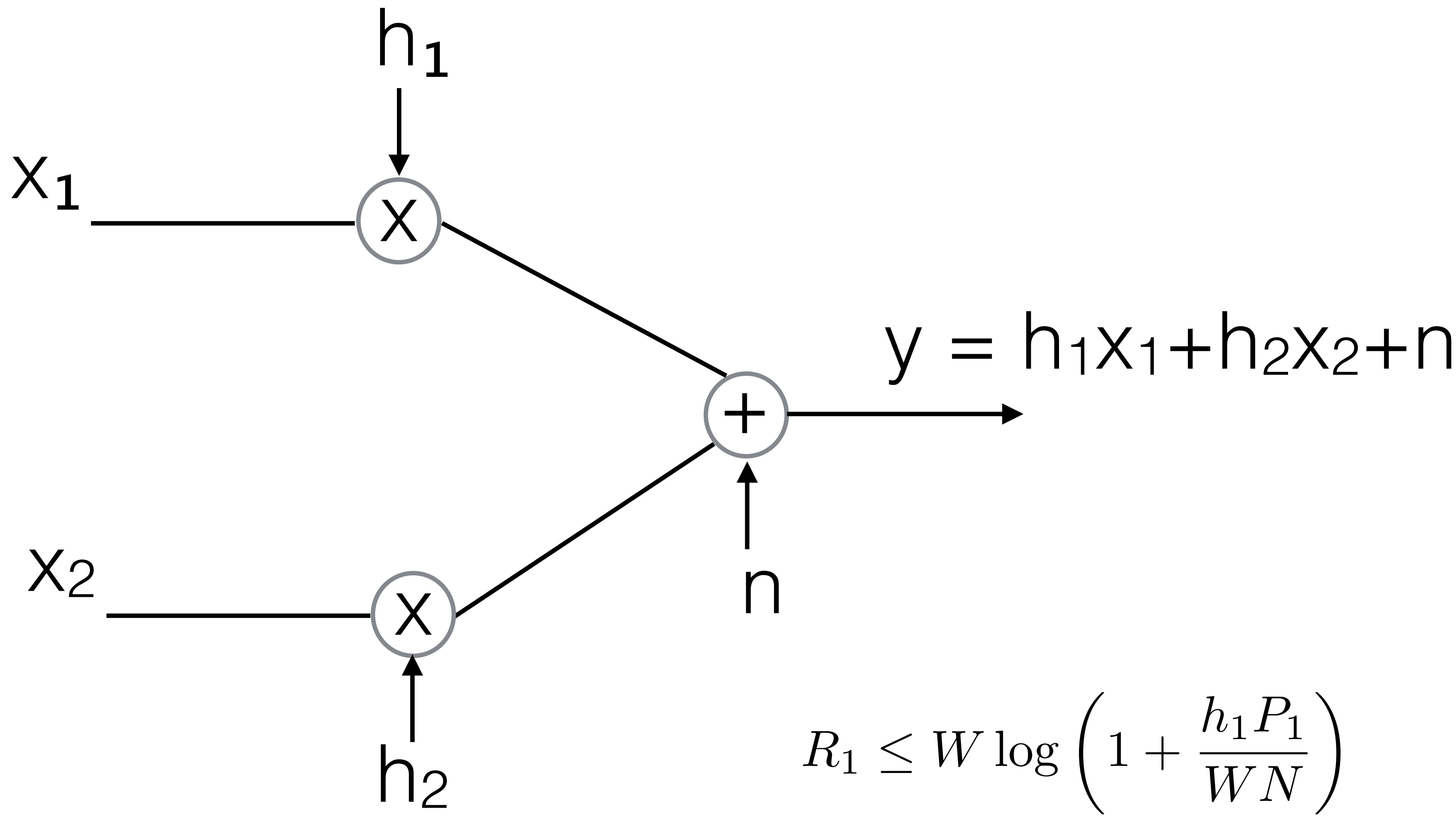

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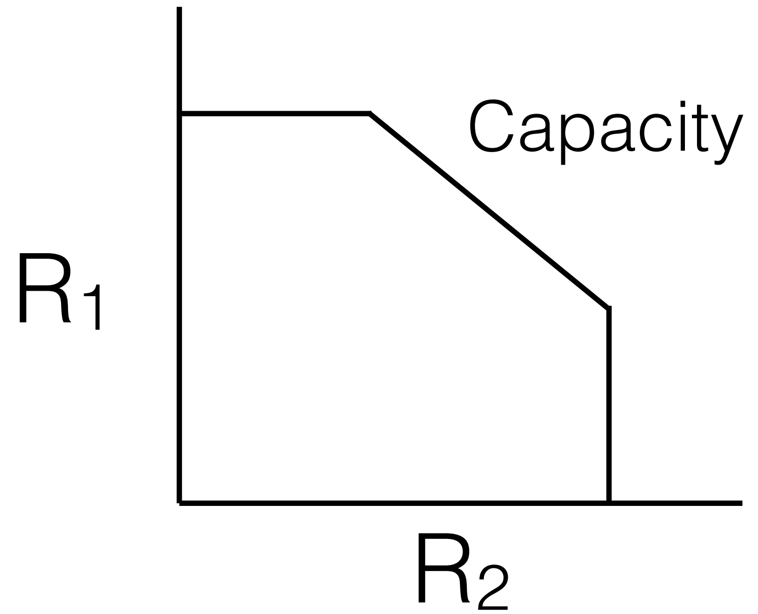

MAC requires Joint/Successive Decoding

FDMA

Instead use FDMA: non-overlapping frequency use

FDMA

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Simple signal processing

FDMA

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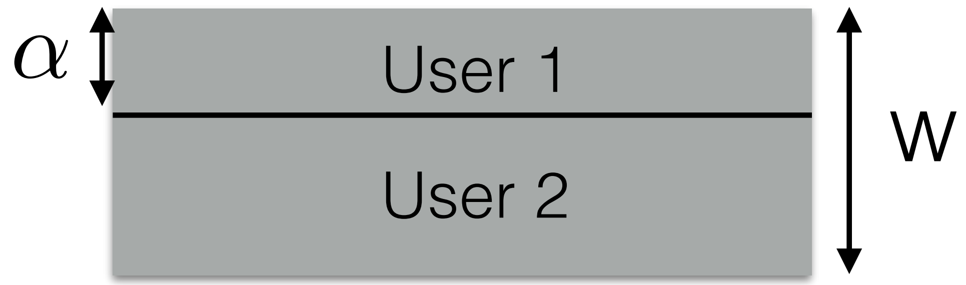
Simple signal processing



FDMA

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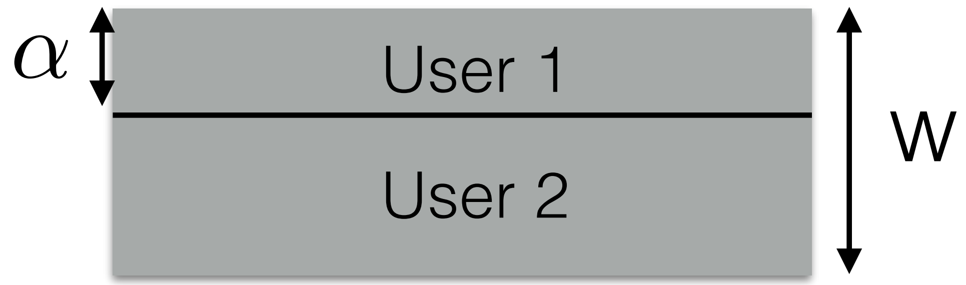

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FDMA

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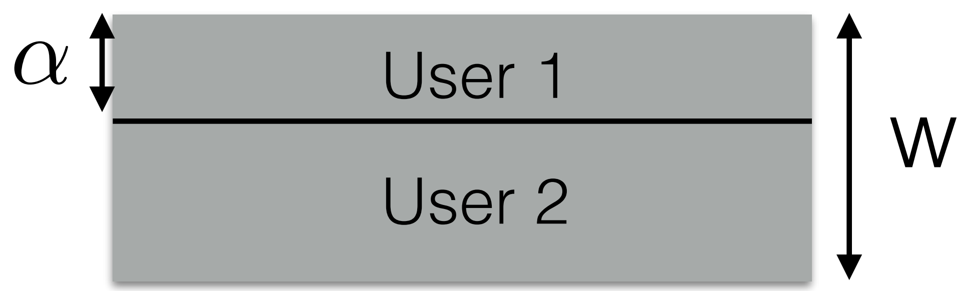

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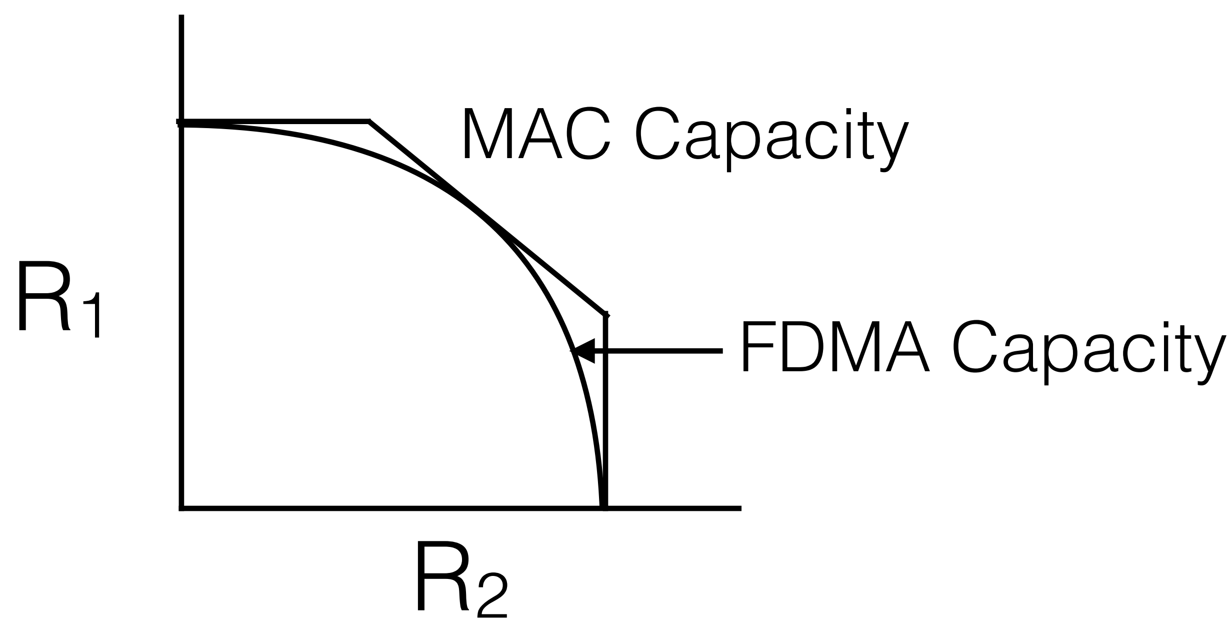
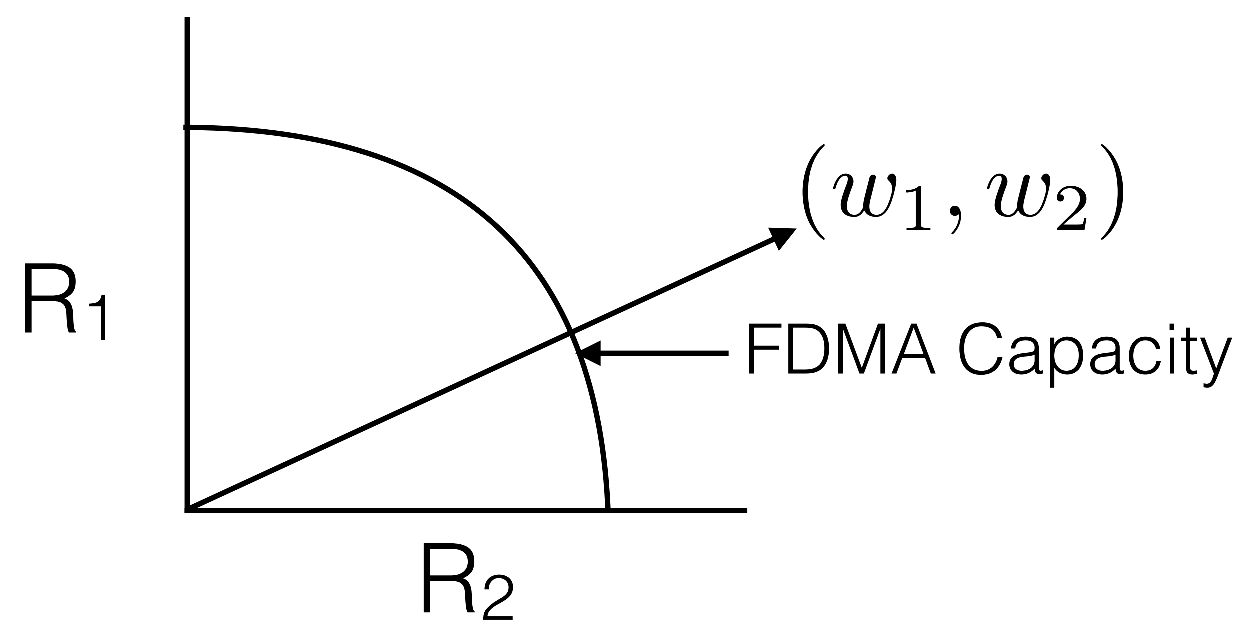


Capacity Region of n Users FDMA

For each vector $(w_1, w_2, \ldots, w_n)$



Capacity Region of n Users FDMA

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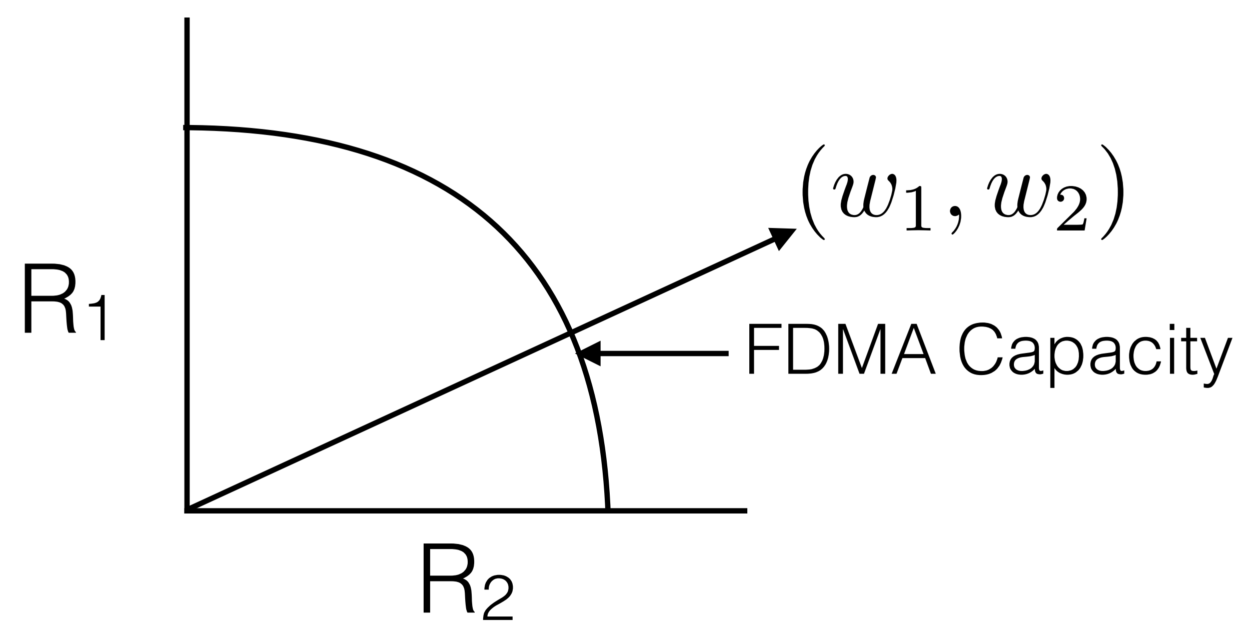
Find opt power $\max_{P_i(f)} \sum_{i=1}^{n} w_i R_i$

non-overlapping freq $P_i(f)P_j(f) = 0$

$f \in W$



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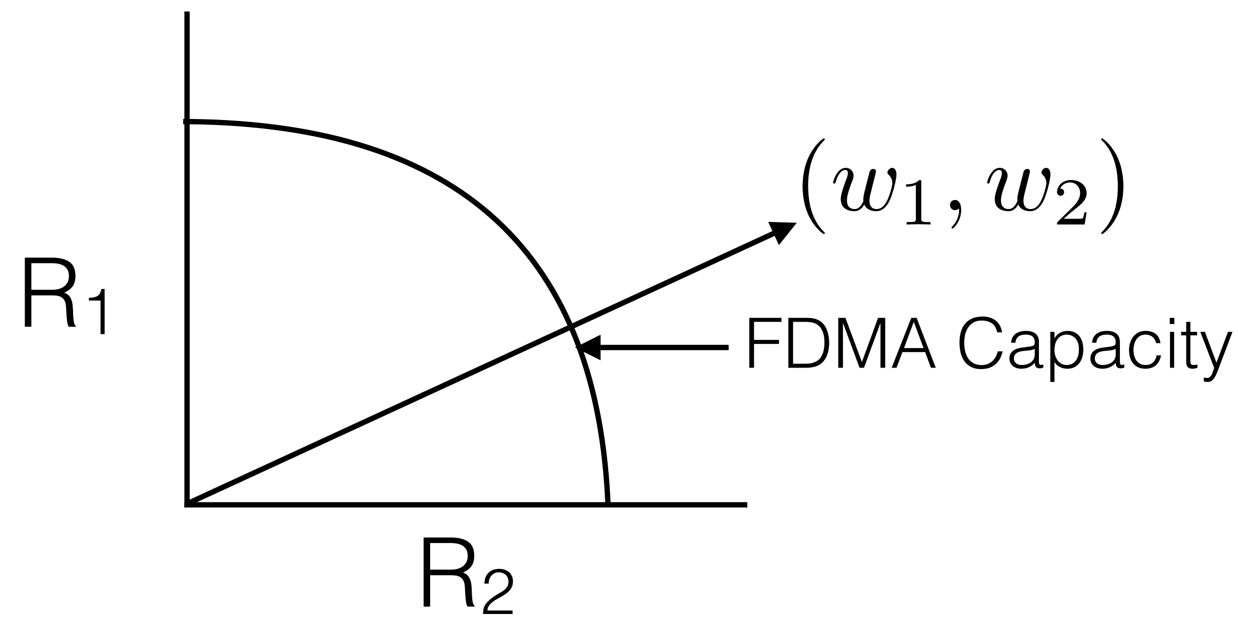
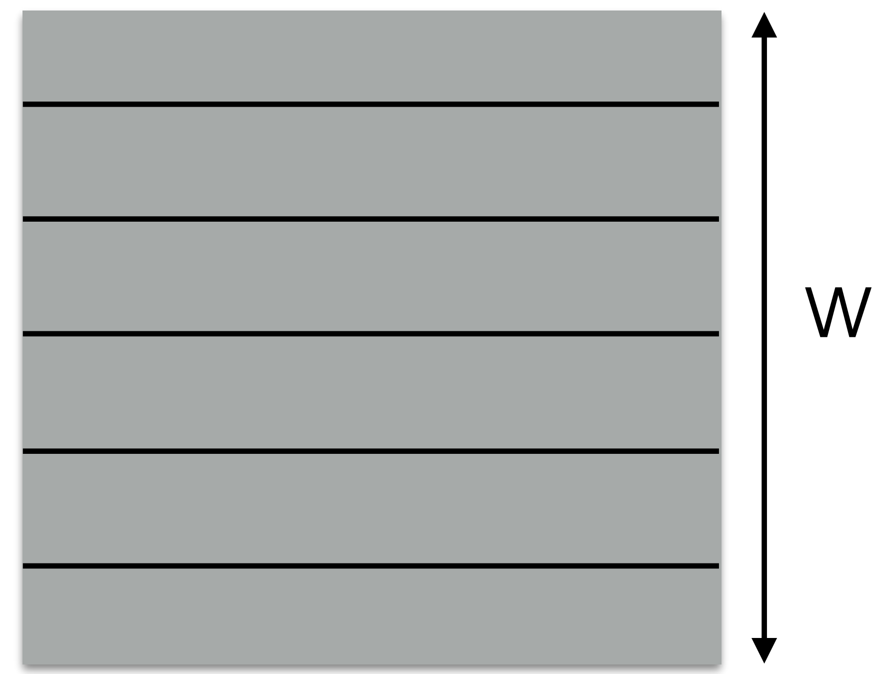


Capacity Region of n Users Discretized FDMA

BW W is partitioned into m bins



Capacity Region of n Users Discretized FDMA

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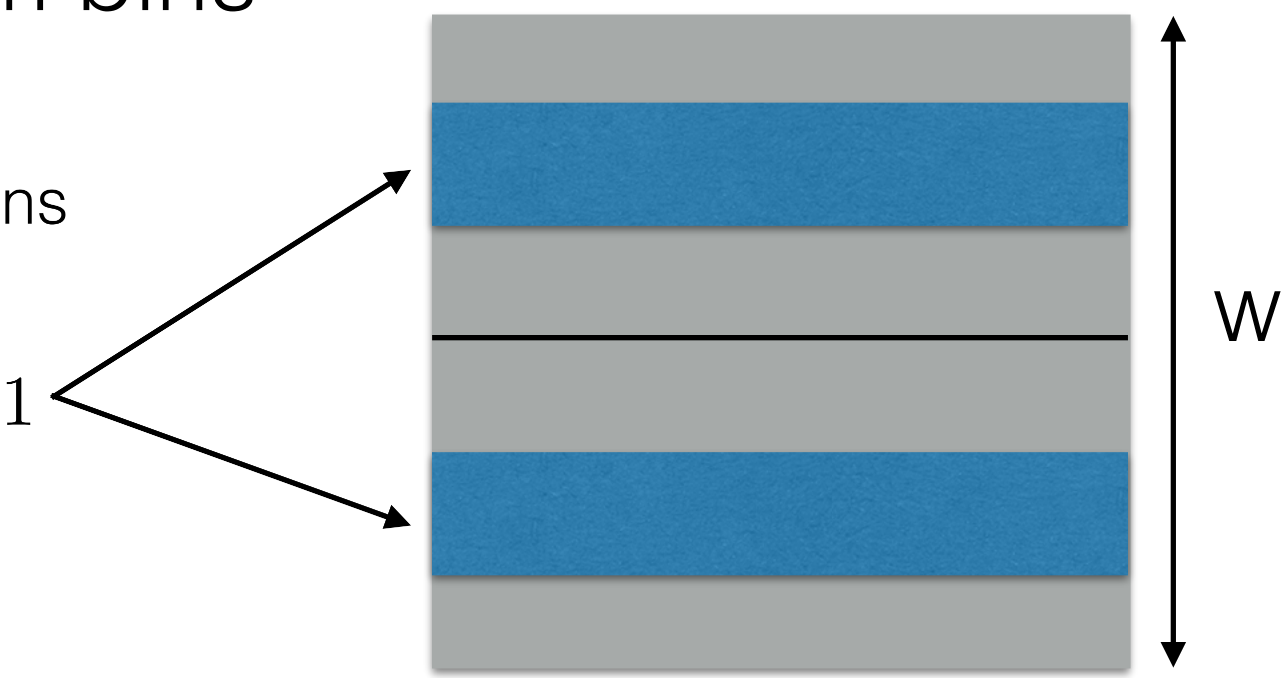
Each user is allocated one or more bins

User i is allocated bin j if $b_{ij} = 1$



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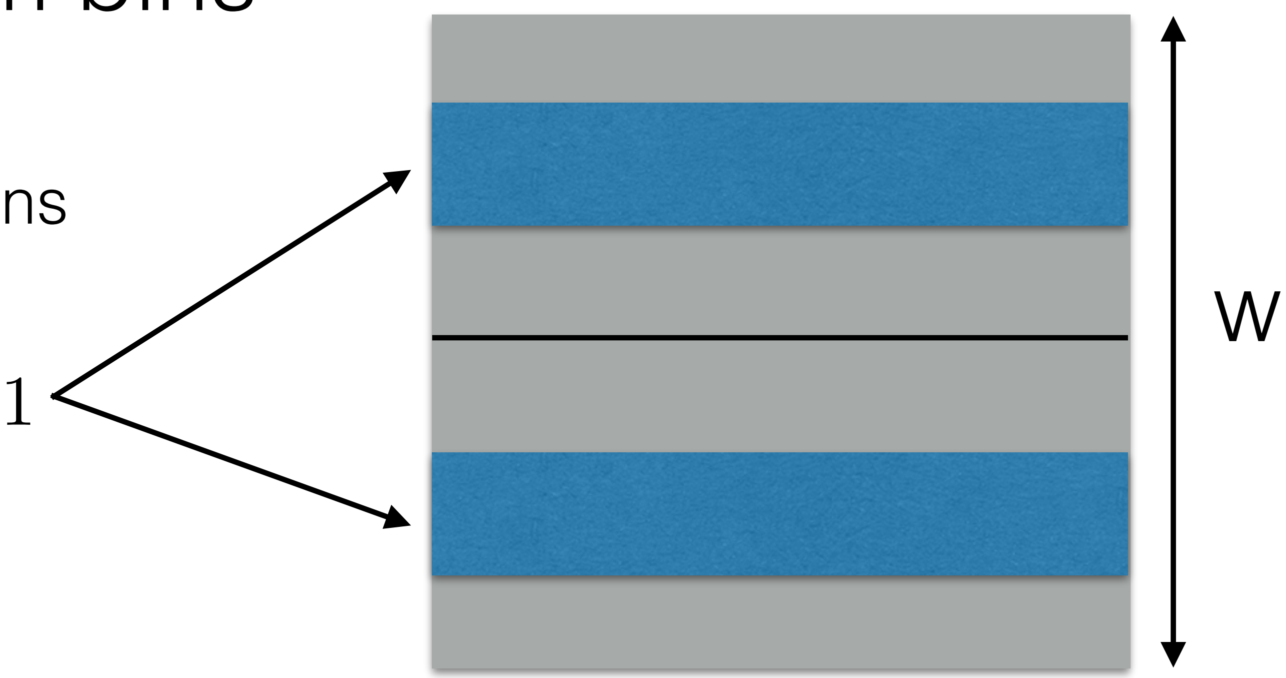
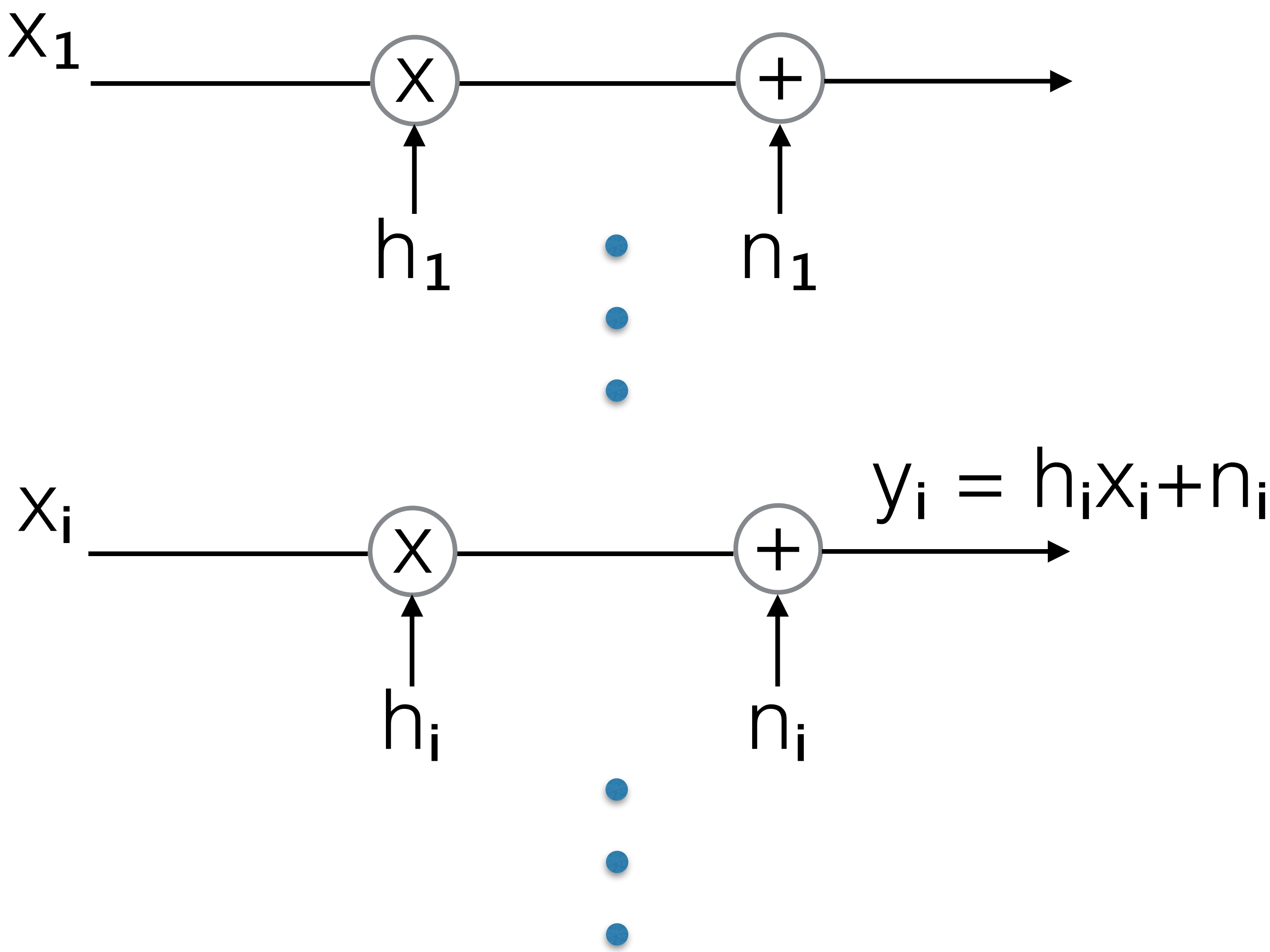


Find bin and power allocation
for each vector $(w_1, w_2, \ldots, w_n)$

$$\max_{P_i, b_{ij} \in \{0,1\}} \sum_{i=1}^{n} w_i \sum_{j=1}^{F_j} b_{ij} R_{ij}$$

Each user sees a Parallel Gaussian Channel



$$r_i = \log\left(1 + \frac{h_i}{n_i} P_i\right)$$

Find powers P_i's to max sum-rate

$$\max_{\sum_{i=1}^{n} P_i \leq P} \sum_{i=1}^{n} \log\left(1 + \frac{h_i}{n_i} P_i\right)$$

Water-filling

Optimal Solution to $\max_{\sum_{i=1}^n P_i \leq P} \sum_{i=1}^n \log\left(1 + \frac{h_i}{n_i} P_i\right)$

is Water-filling

$$P_i^\star = \left(\mu - \frac{n_i}{h_i}\right)^+ \qquad \sum_{i=1}^n P_i^\star = P$$

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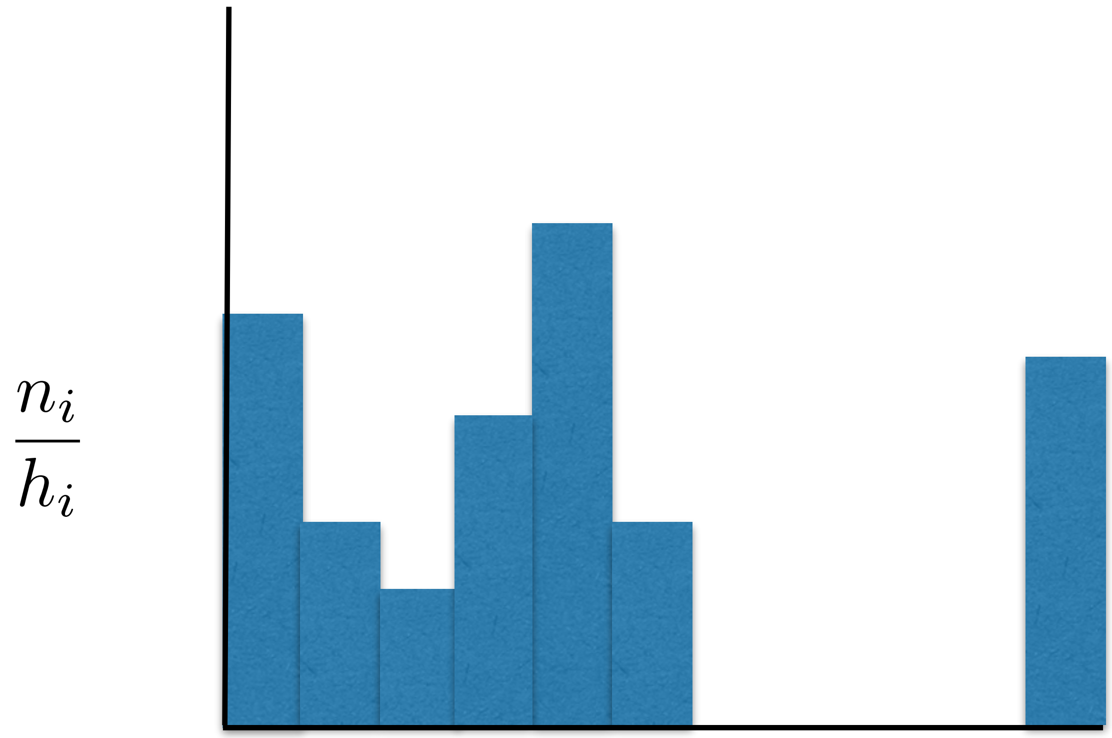

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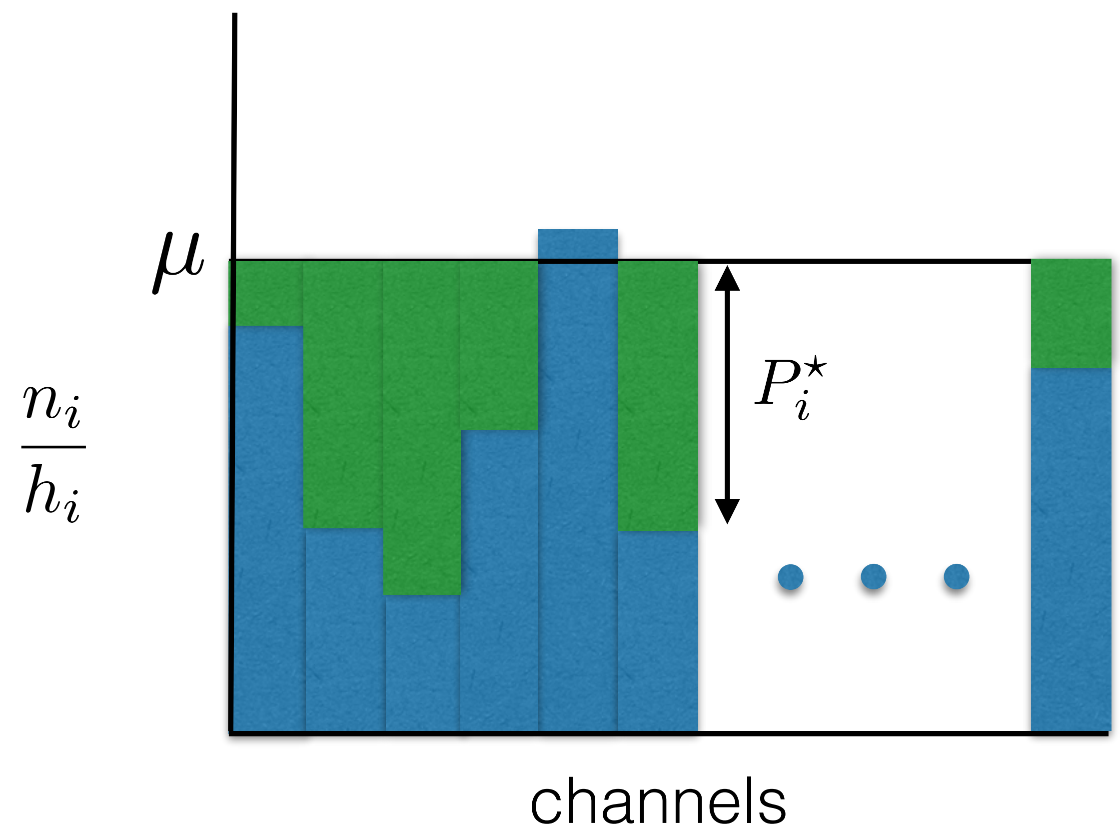
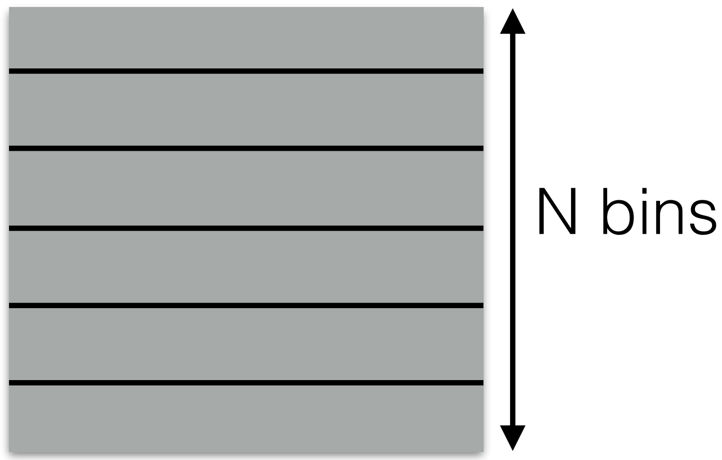
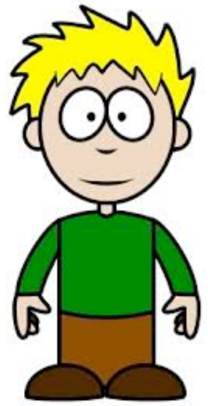
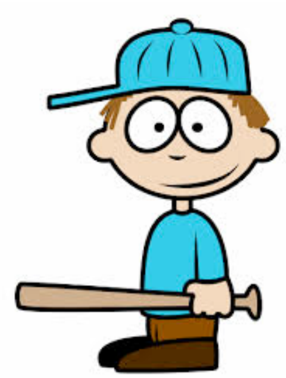


OFDMA Capacity as Partitioning Problem



sum-rate with water-filling





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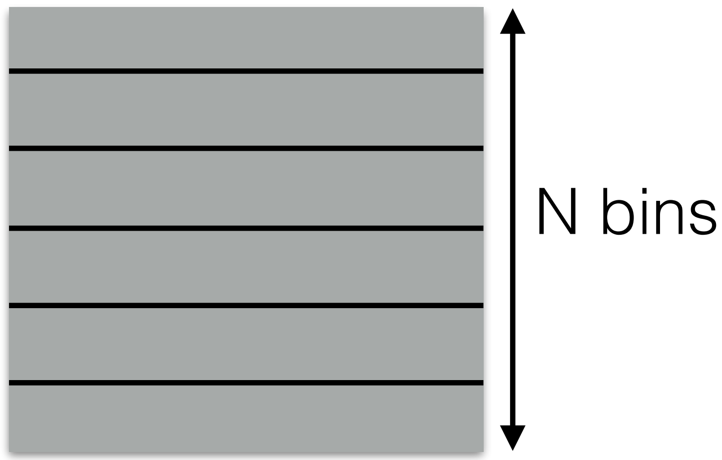
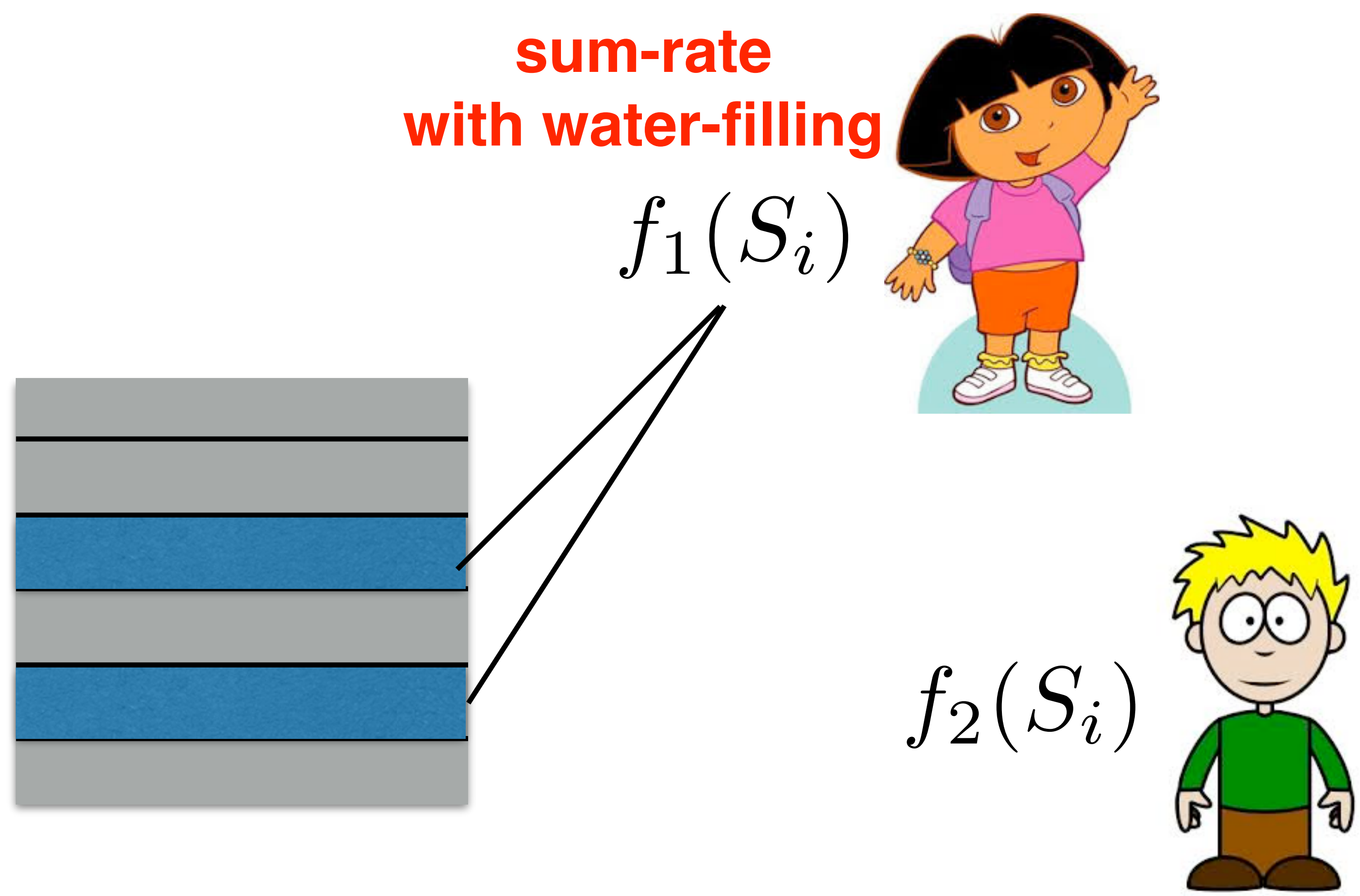
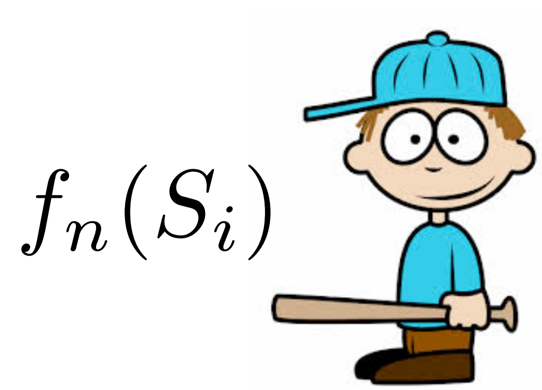




Partition N bins

into $S_1, S_2, \ldots, S_n$



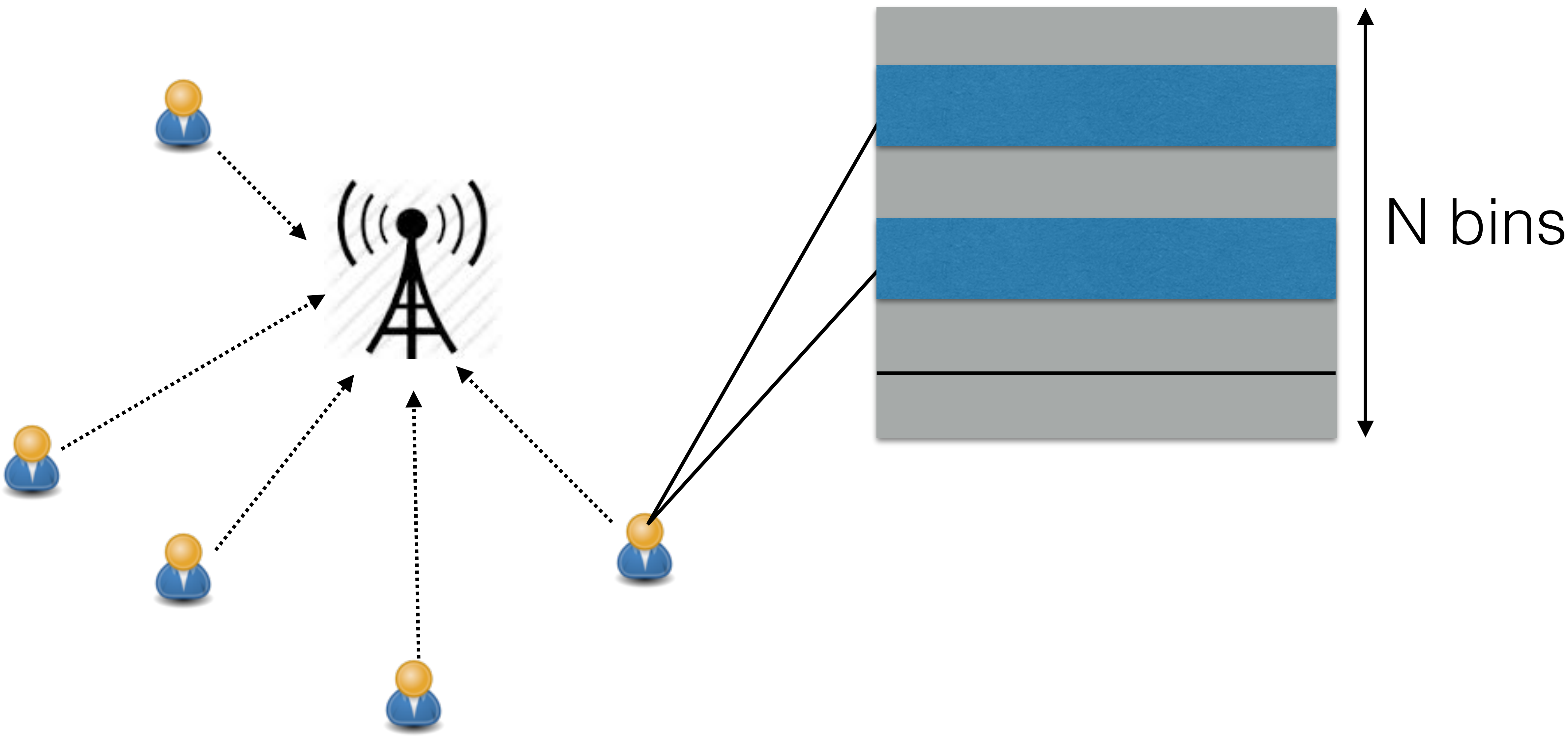
Known Results

Heuristics

- Convex Relaxation (YuCioffi'02)
- KKT (KimHanKim'05)

Subcarrier and Power allocation in uplink OFDMA

Identical to finding OFDMA capacity



Given sub-carrier (bins) allocation, find power using water-filling to max sum-rate

1/2 Approx.

Use Greedy Algorithm

- check if each users incentive $\mathbf{f}_i$ (sum-rate with water-filling) is
  - sub-modular
  - monotone

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Use Greedy Algorithm

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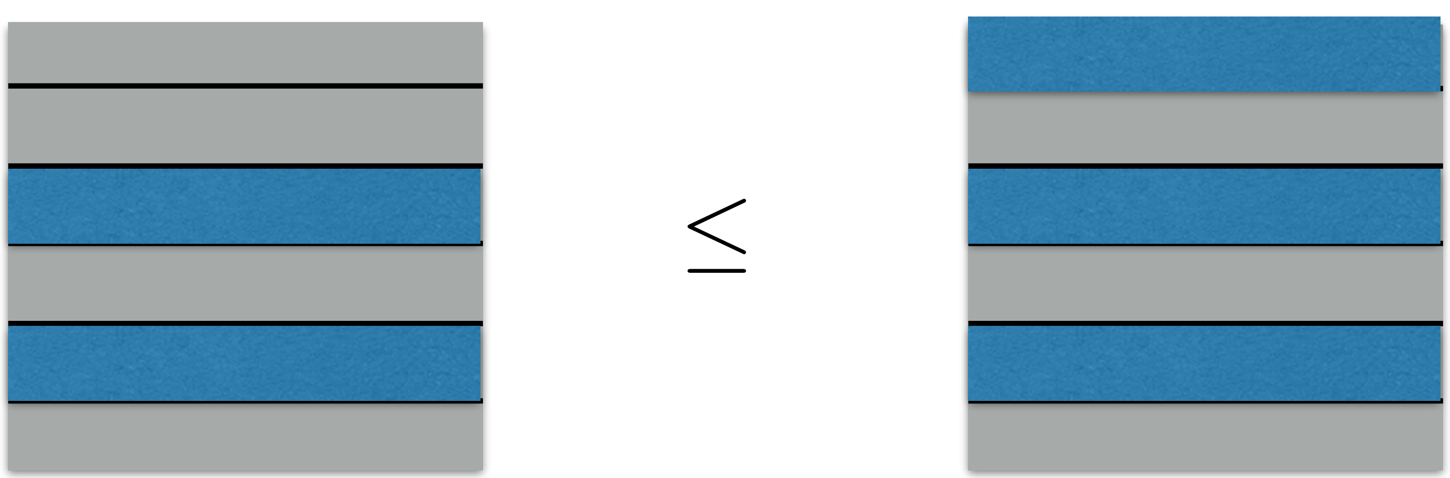
Monotonicity is clear : More channels give larger rate

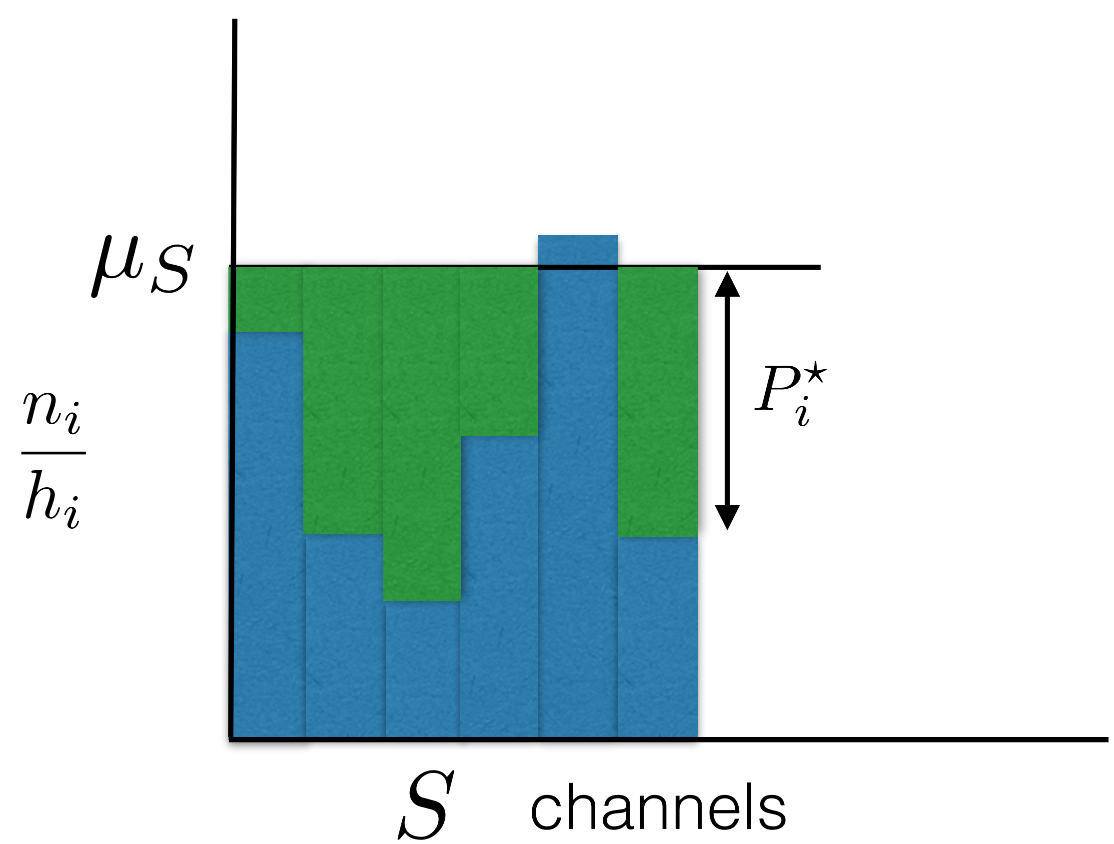


Water-filling as a Set-Function

Let S be a set of channels then sum-rate is

$$R(S) = \max_{\sum_{i=1}^{n} P_i \leq P} \sum_{i \in S} \log\left(1 + \frac{h_i}{n_i} P_i\right)$$

Water-filling $\quad P_i^\star = \left(\mu_S - \frac{n_i}{h_i}\right) \qquad \sum_{i \in S} P_i^\star = P$



Sub-Modularity of Water-Filling

To check $\quad R(S \cup \{i\}) - R(S) \geq R(T \cup \{i\}) - R(T)$

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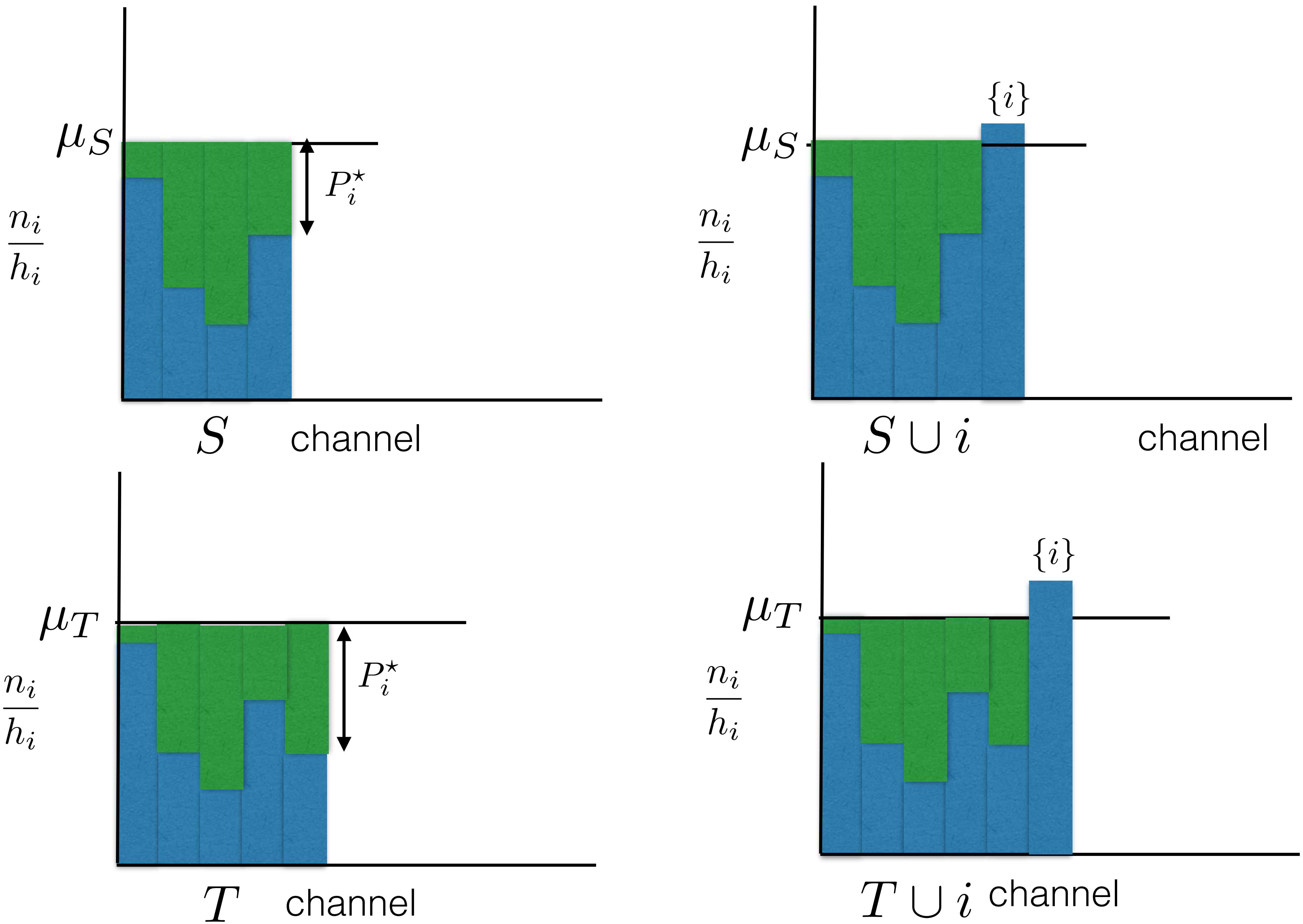

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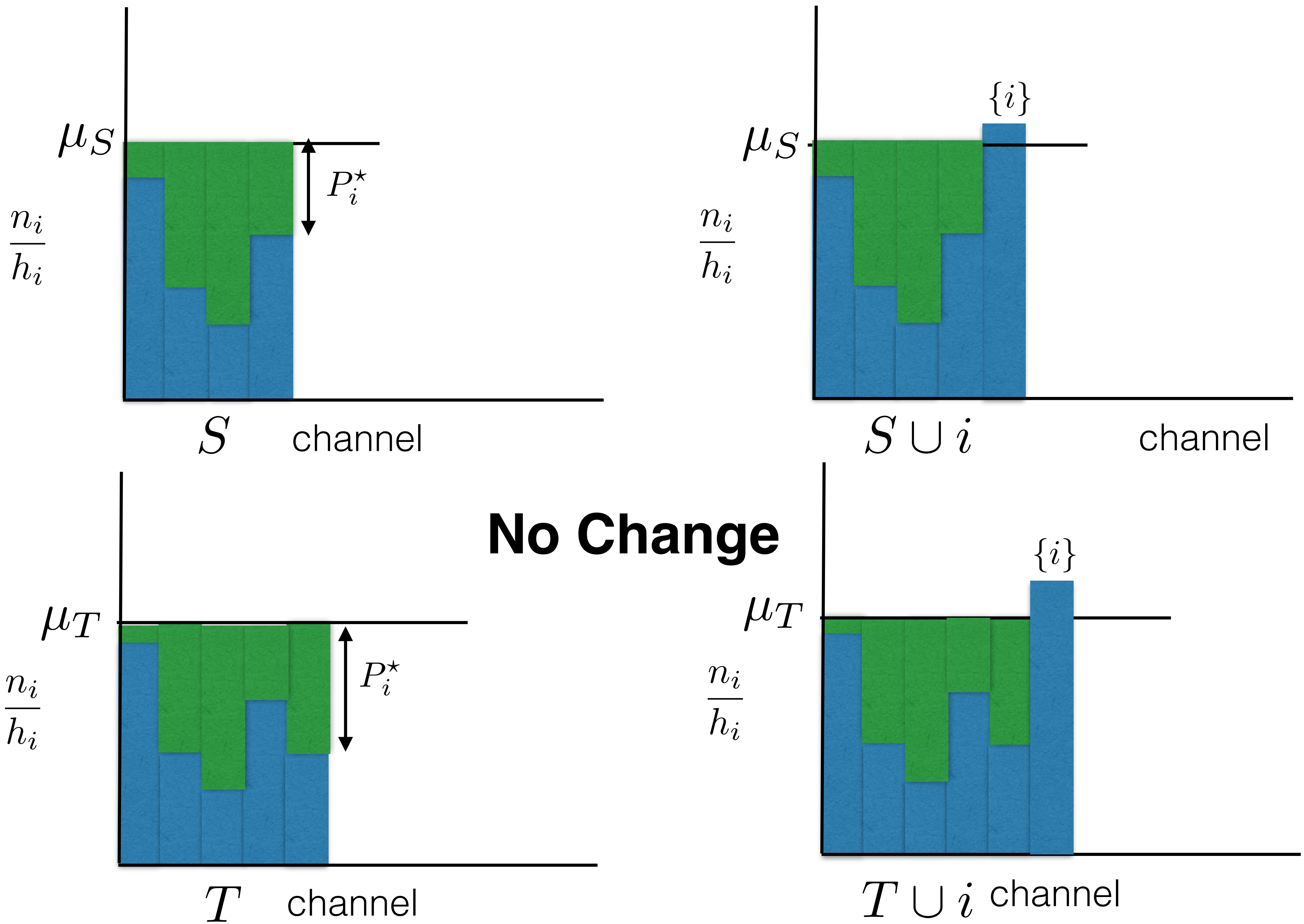

In general there is no easy way

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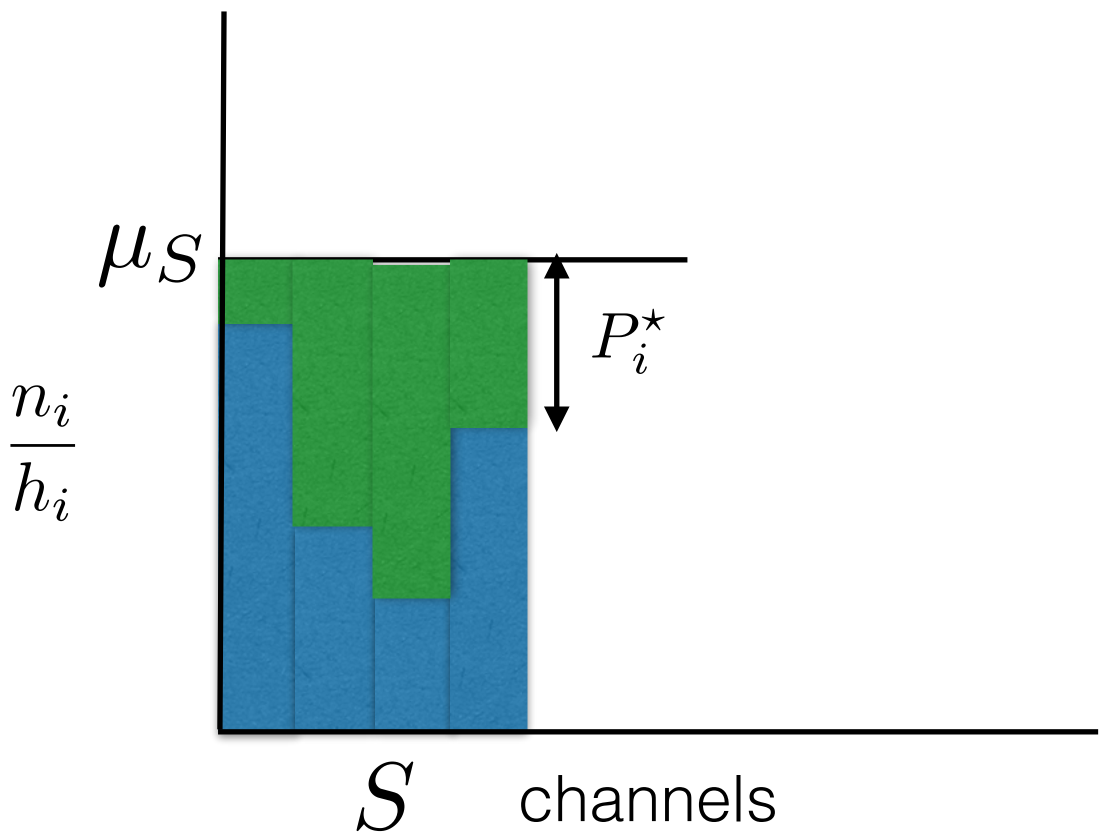

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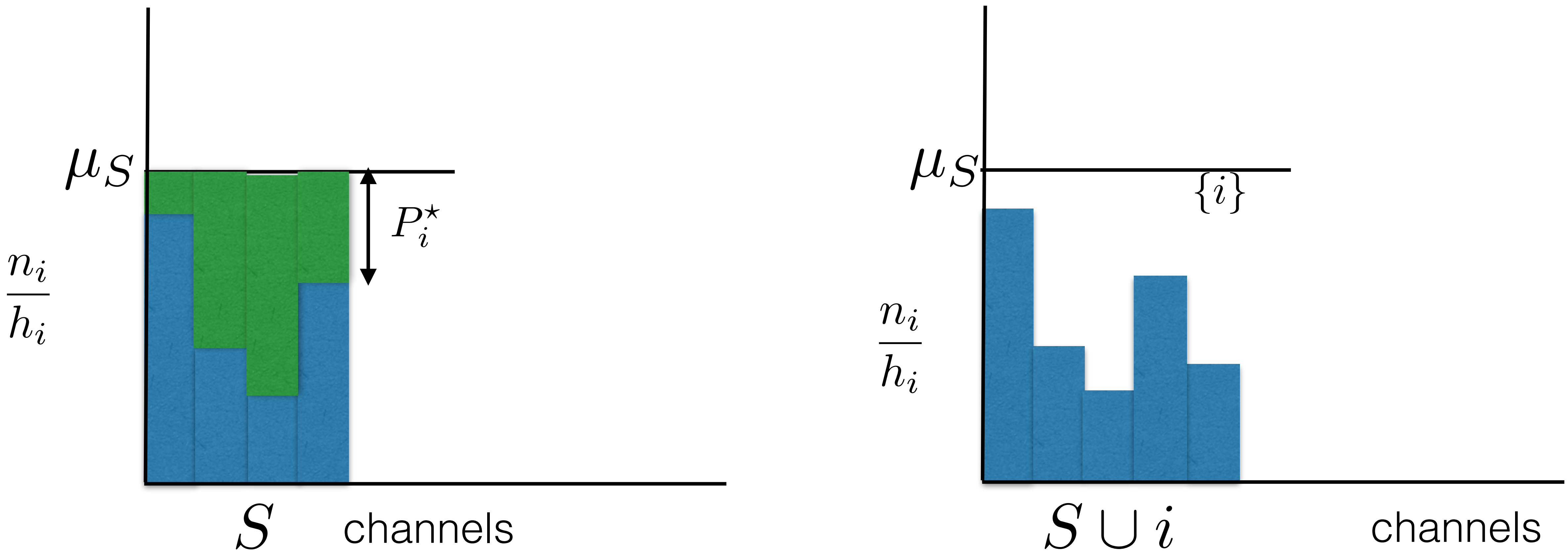

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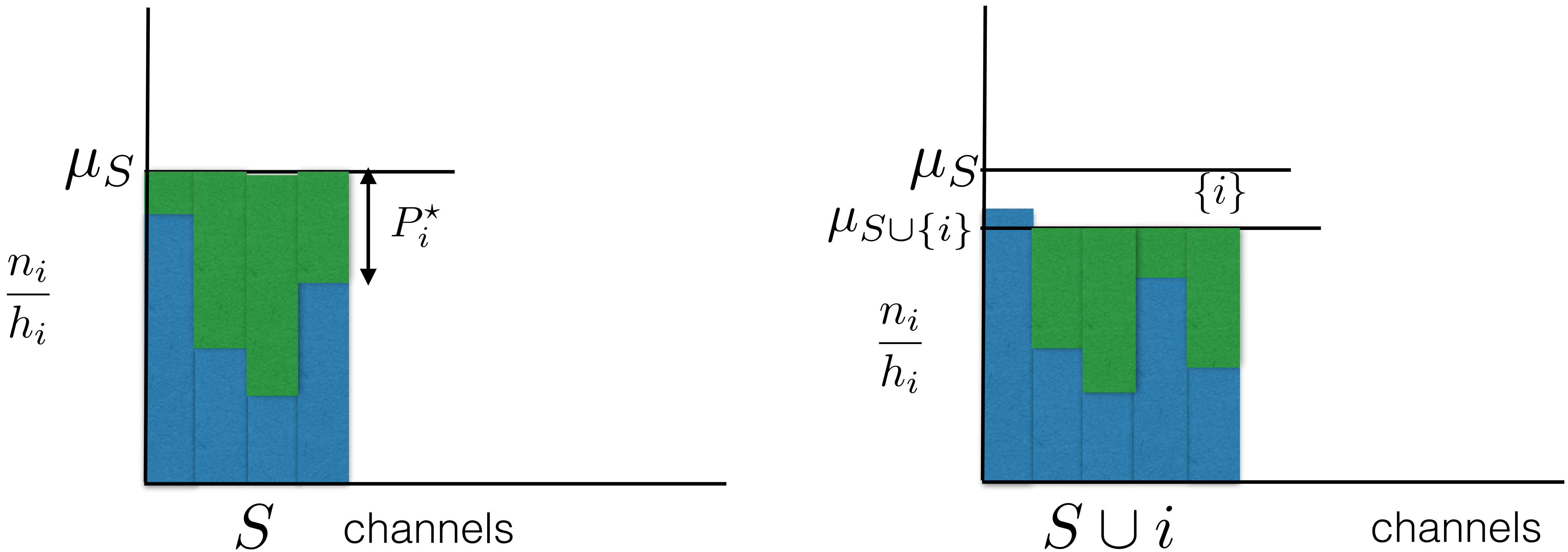

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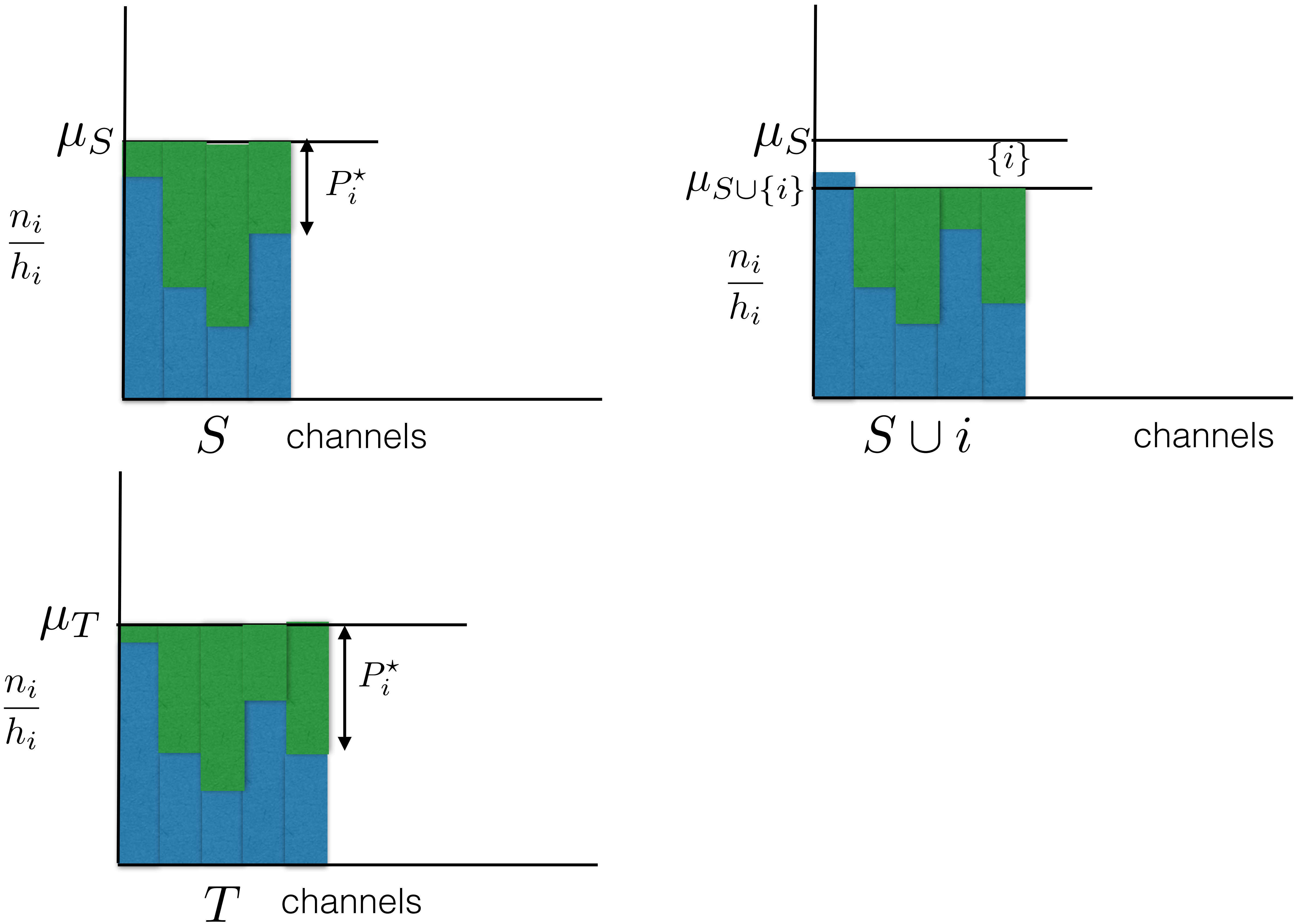

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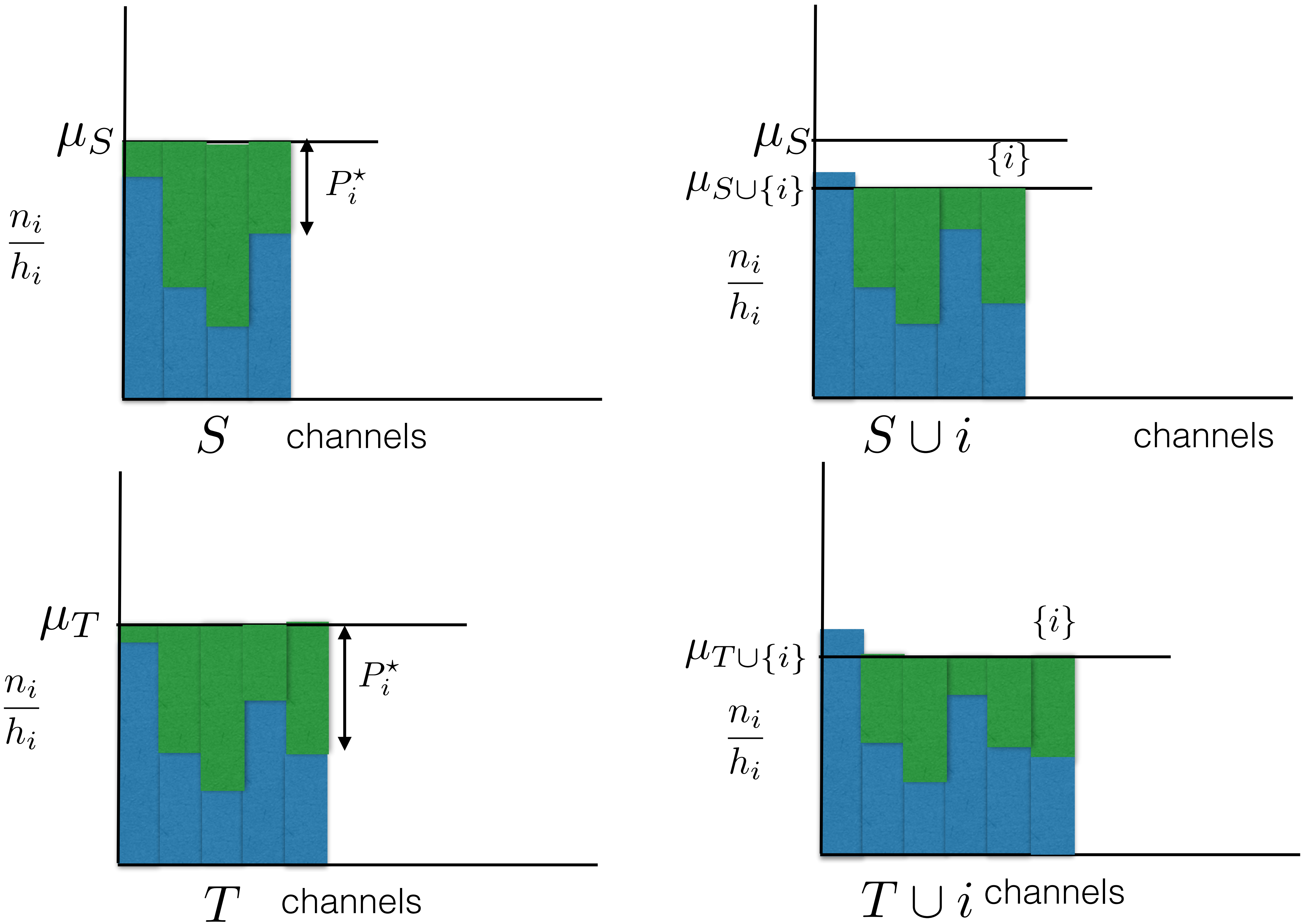

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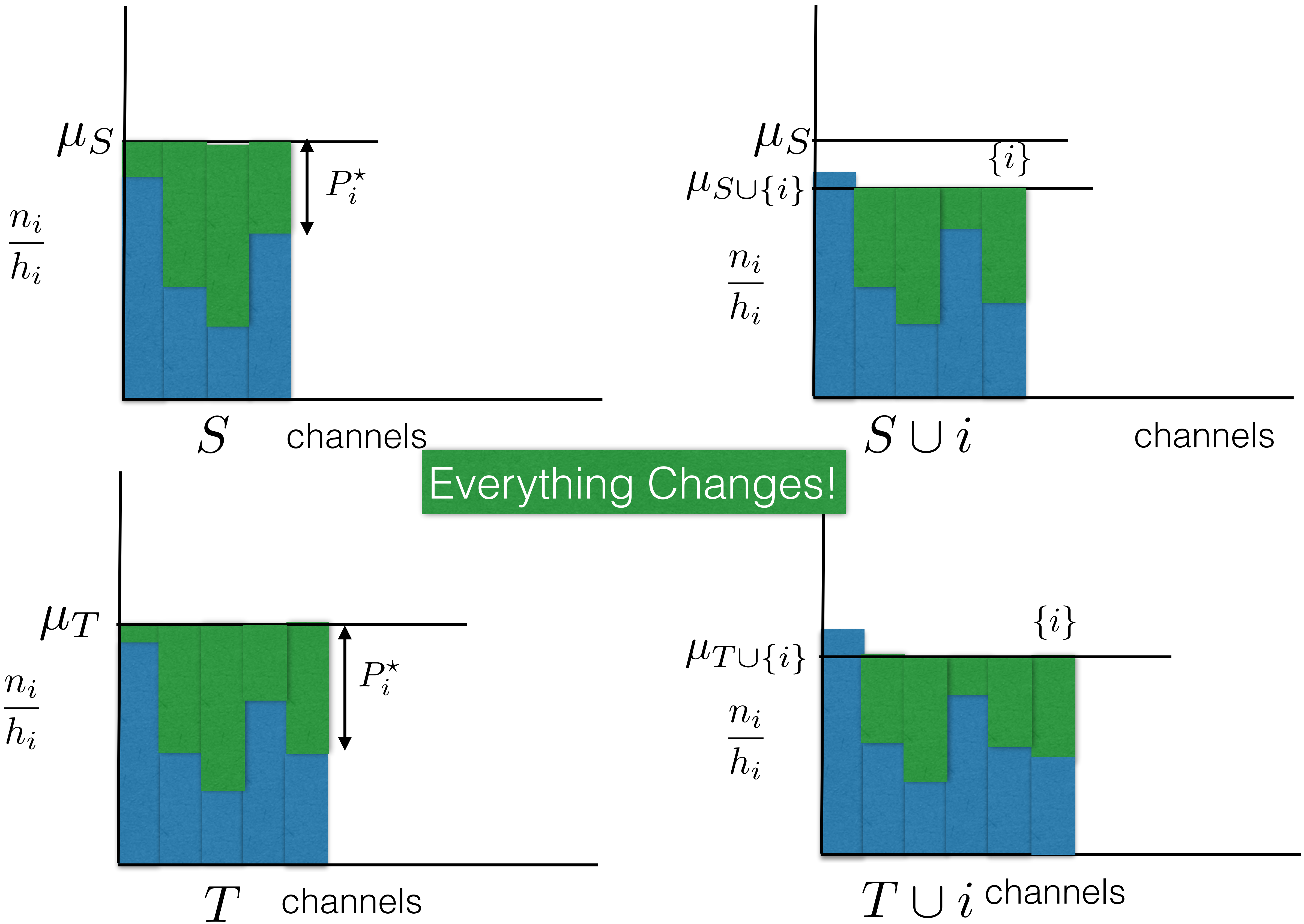

Idea

A. Majorization

B. Karamata's Inequality

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Two vectors **a** and **b** arranged in descending order

$$\mathbf{a} = [a_1 a_2 \ldots a_n] \qquad \mathbf{b} = [b_1 b_2 \ldots b_n]$$

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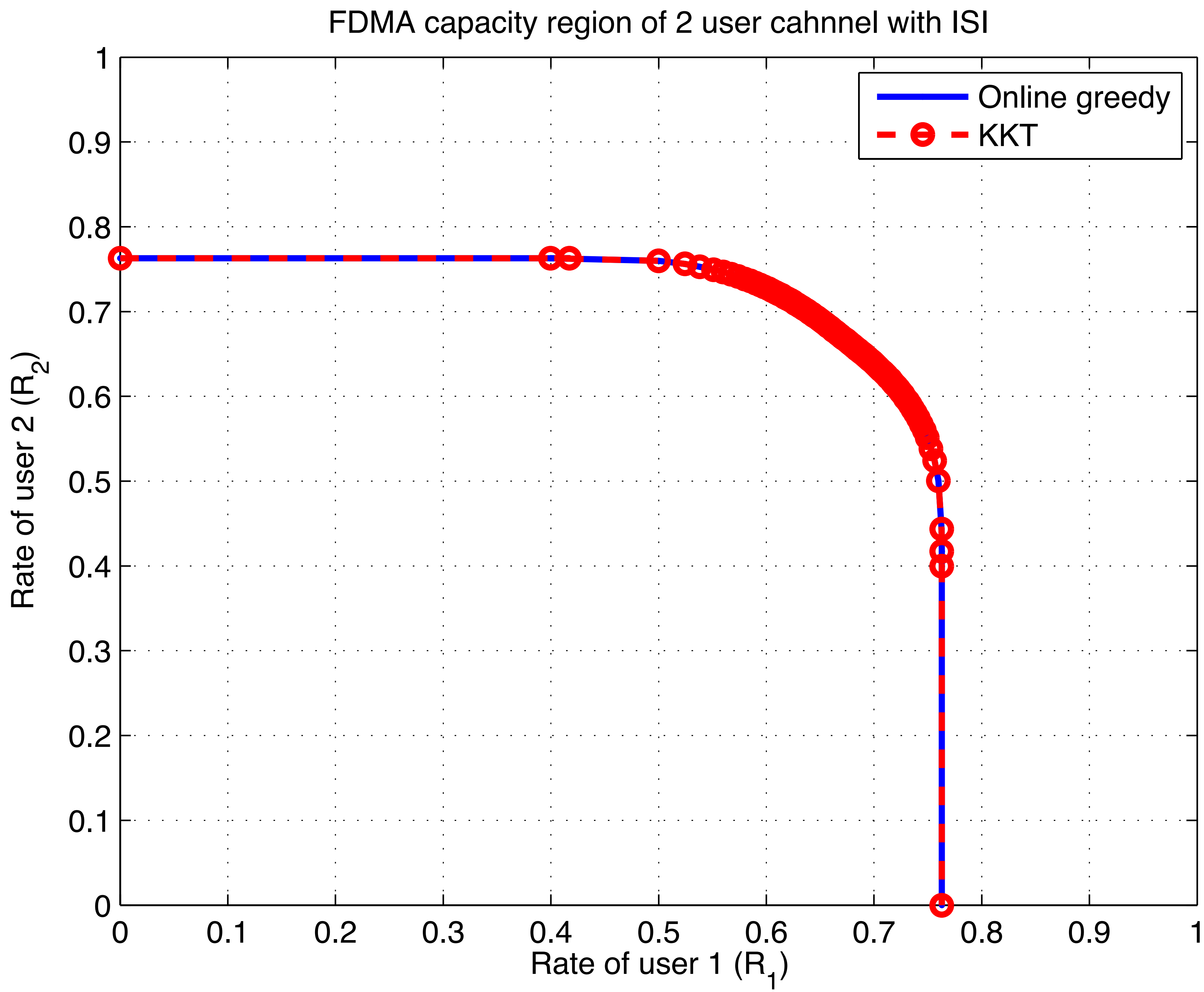
Proof very specific to *log* utility

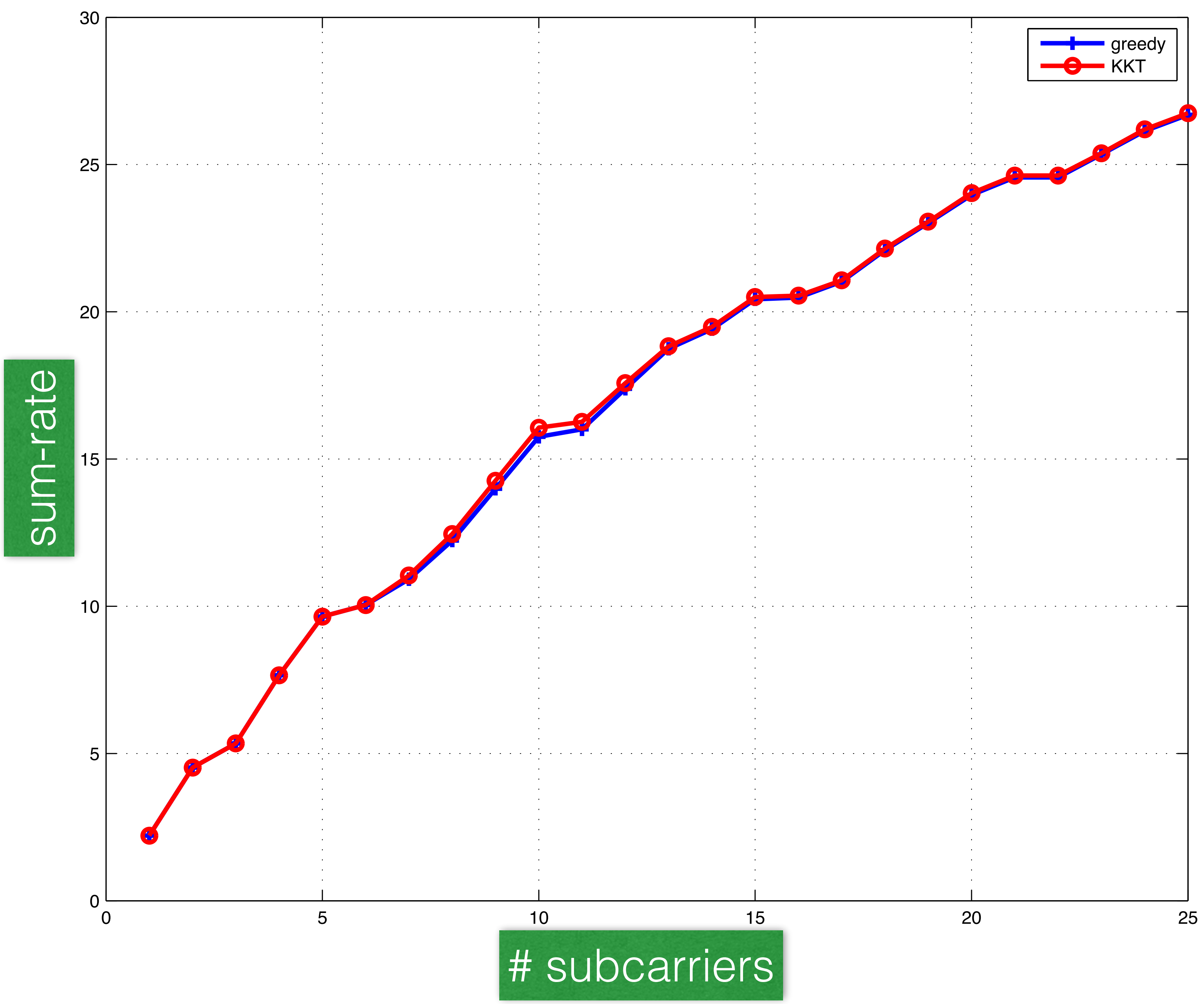
Simulations: Capacity

Comparison with Heuristics



Simulations: Sum-rate

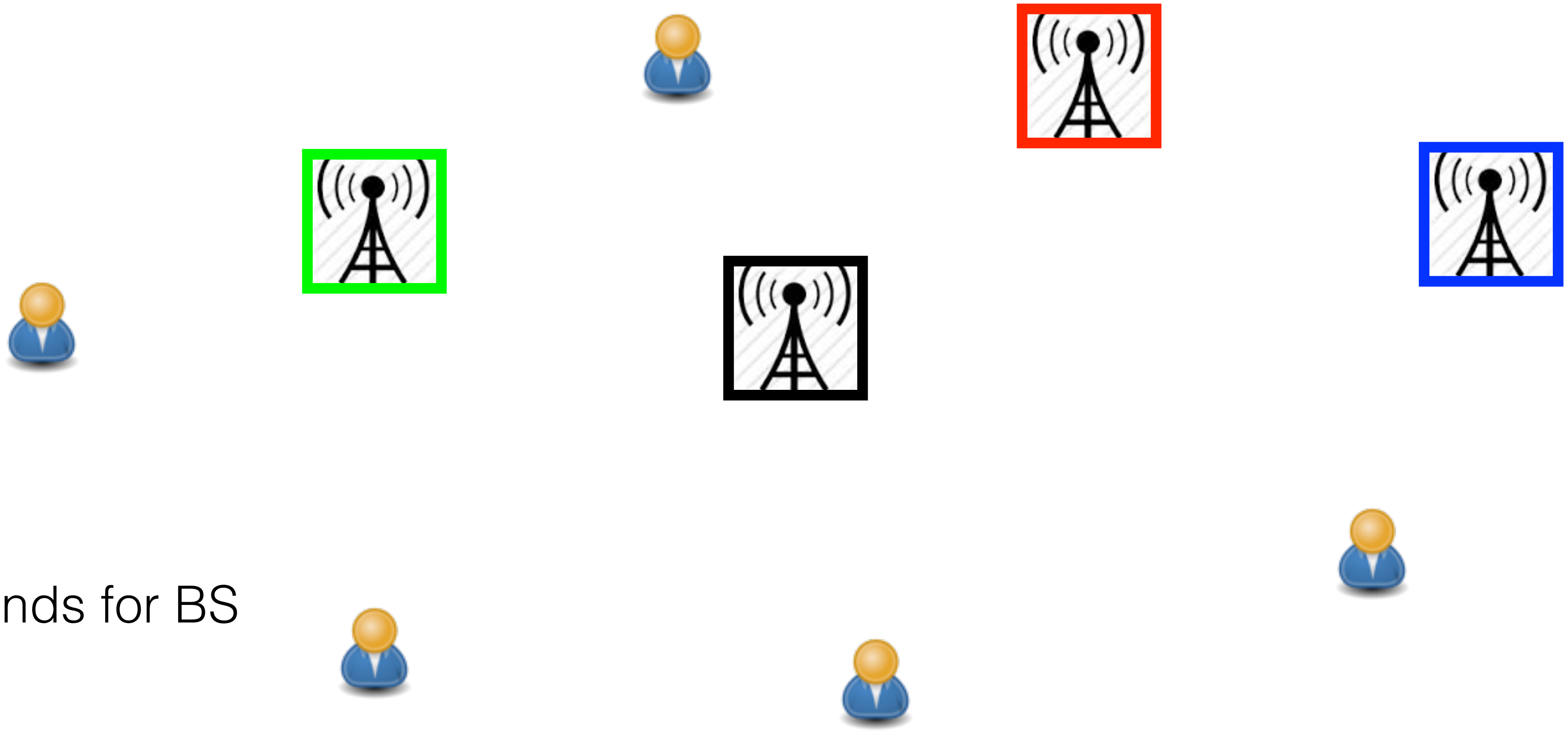
Comparison with Heuristics with 10 users



What more !

Downlink BS association

Associate each mobile to one BS



Disjoint bands for BS

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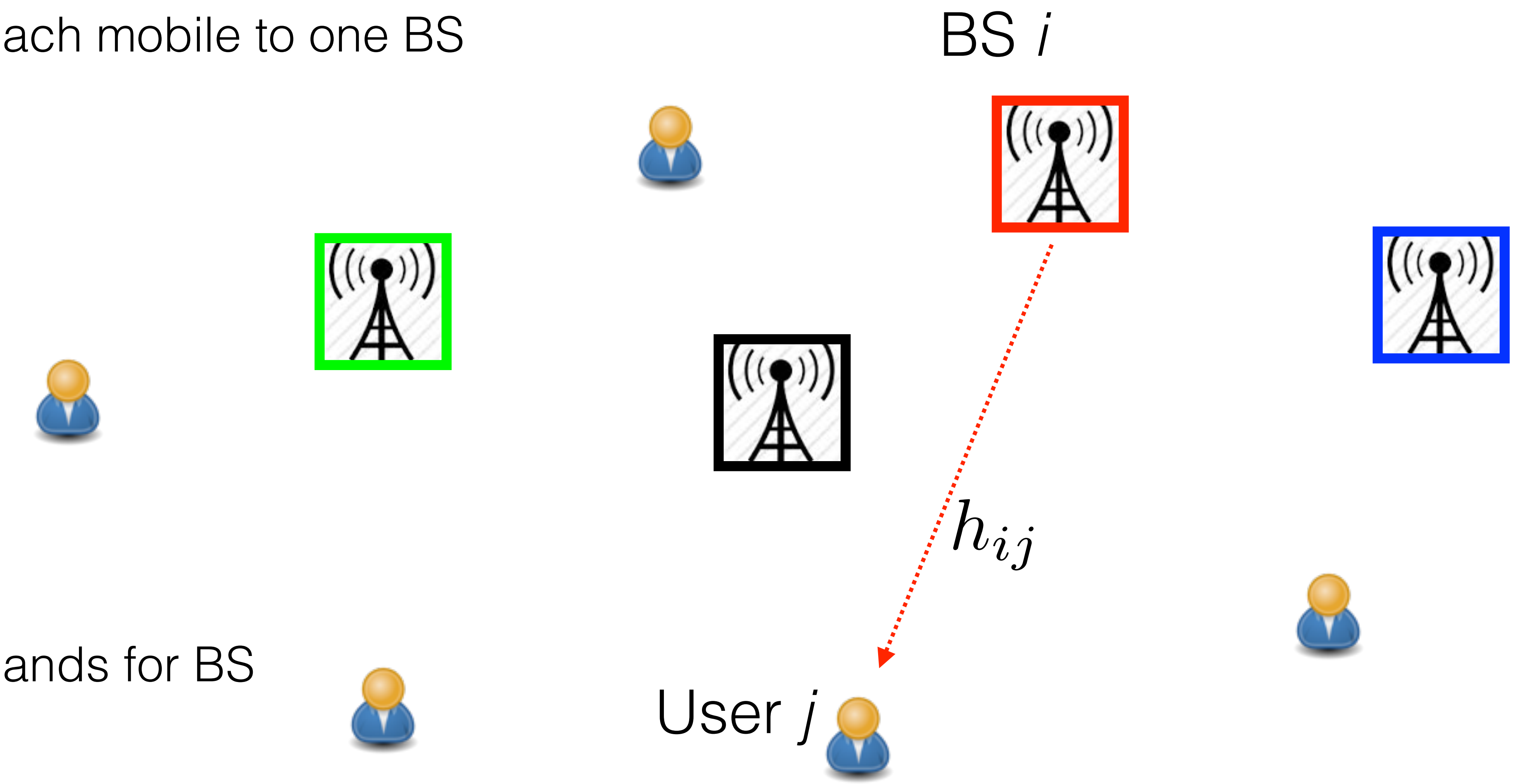

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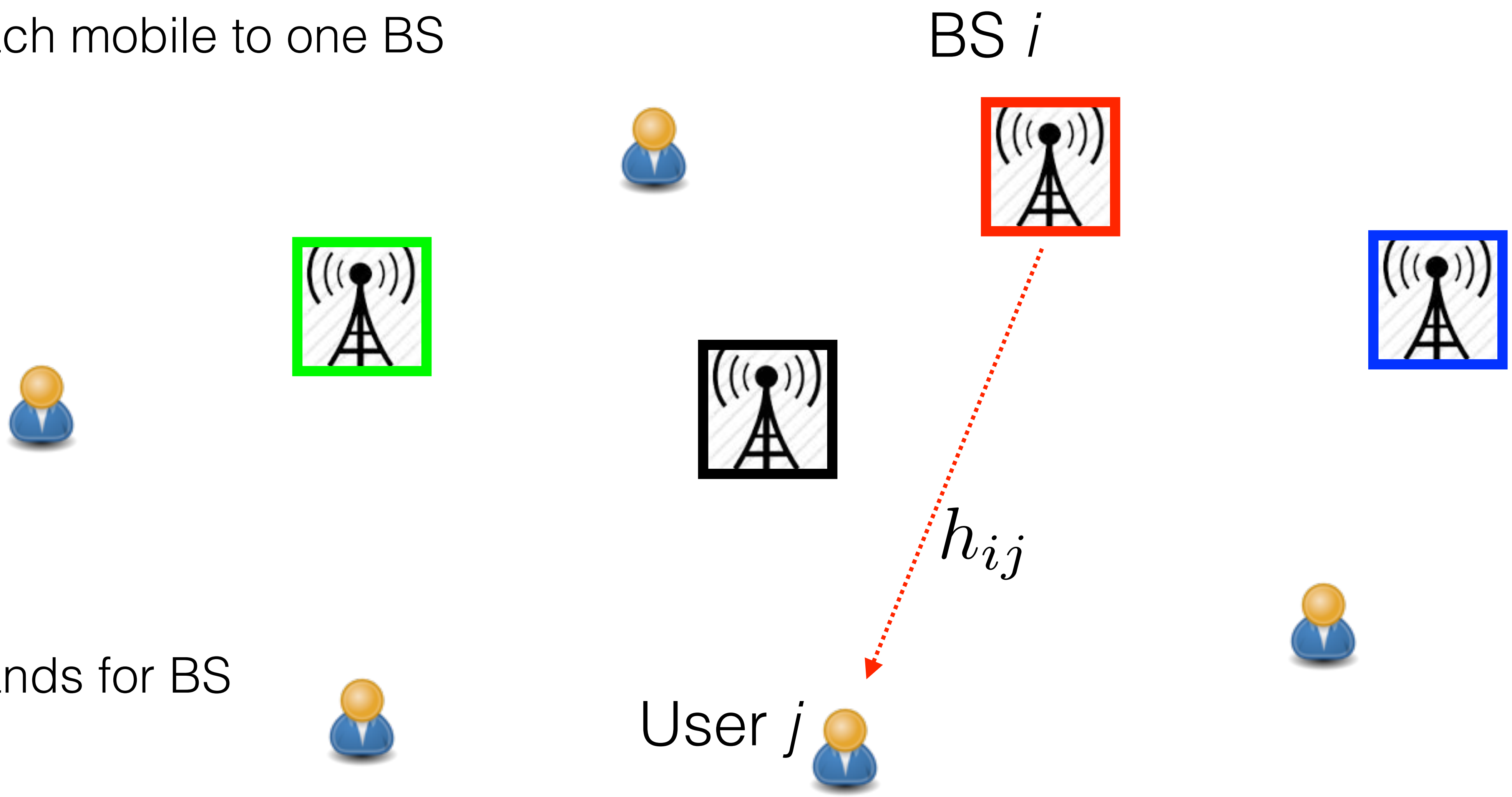

Disjoint bands for BS

Each BS gives power according to log utility for all associated users

$$R_i = \max_{\sum_{j \in A_i} P_j \leq P} \sum_{j \in A_i} \log(1 + P_j h_{ij})$$

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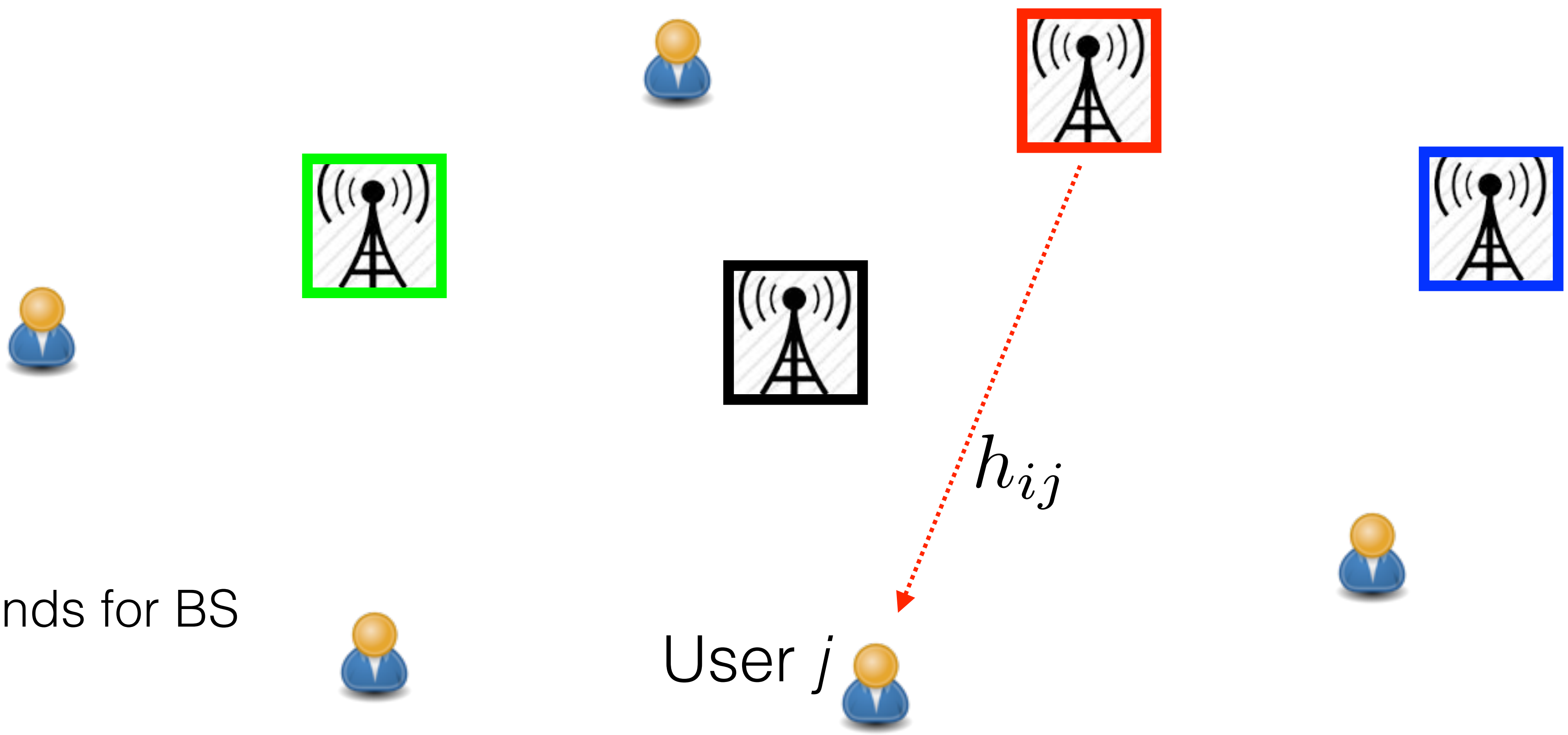

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Find association to maximize the sum-rate

$$\max_{A_i} \sum_{i=1}^{m} R_i$$

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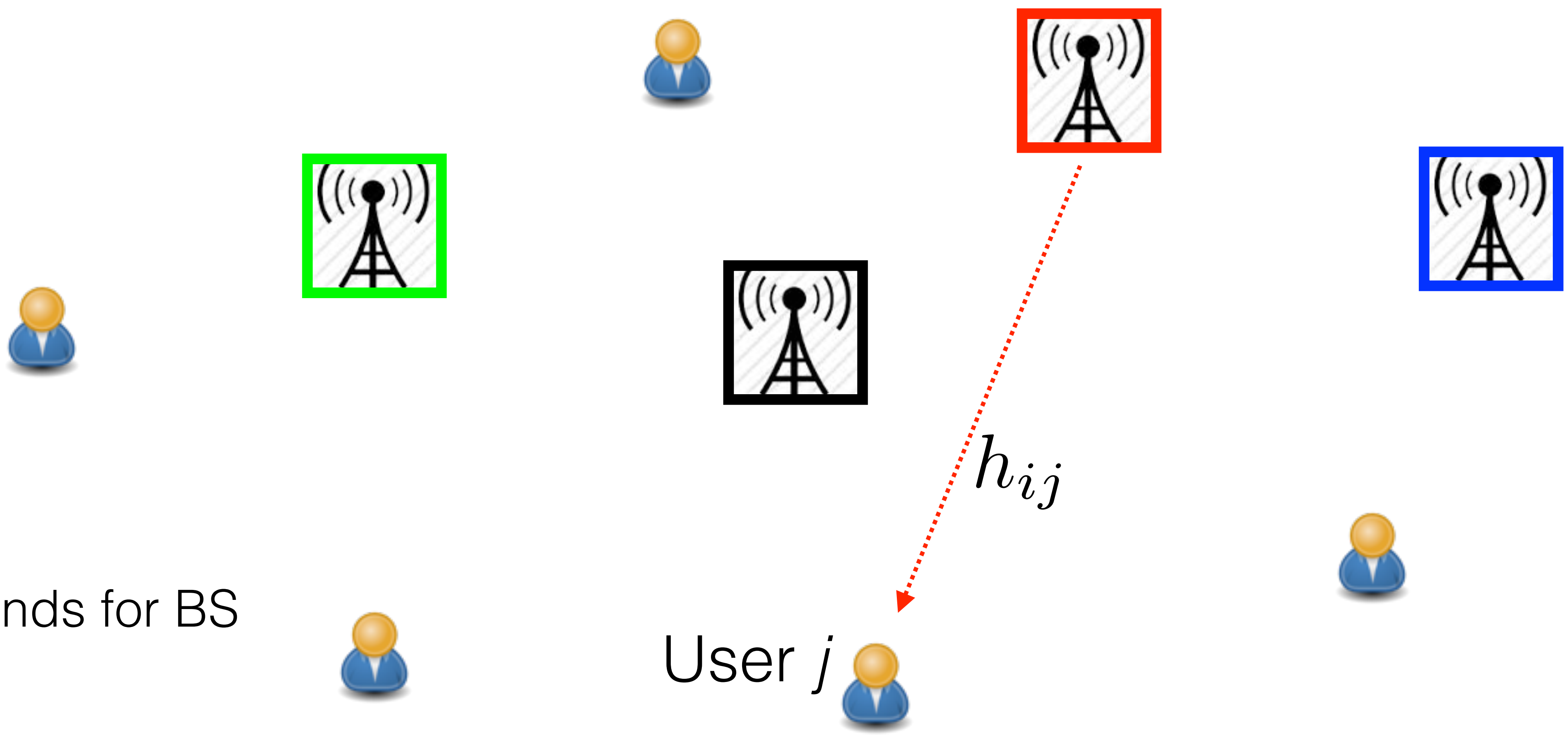

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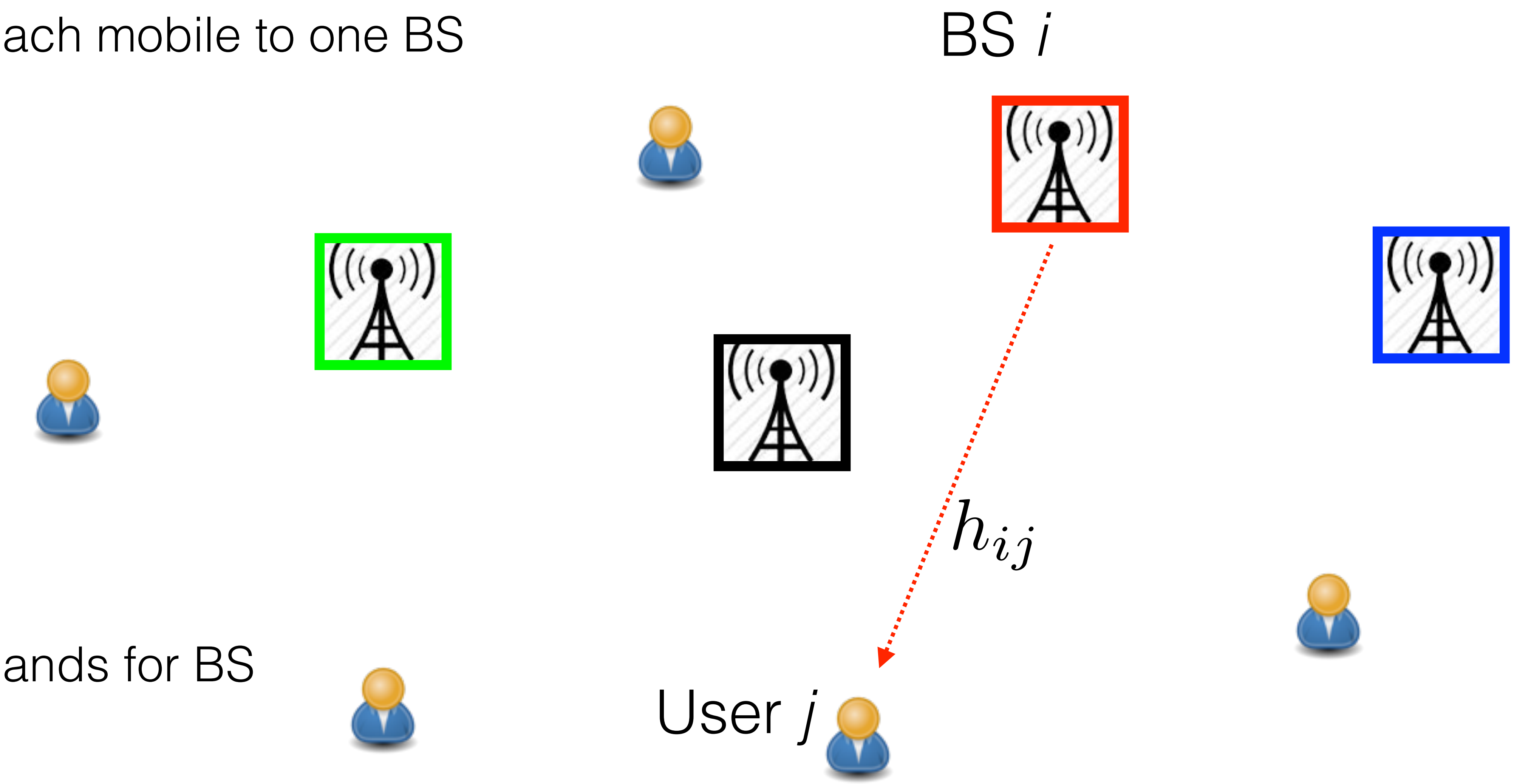

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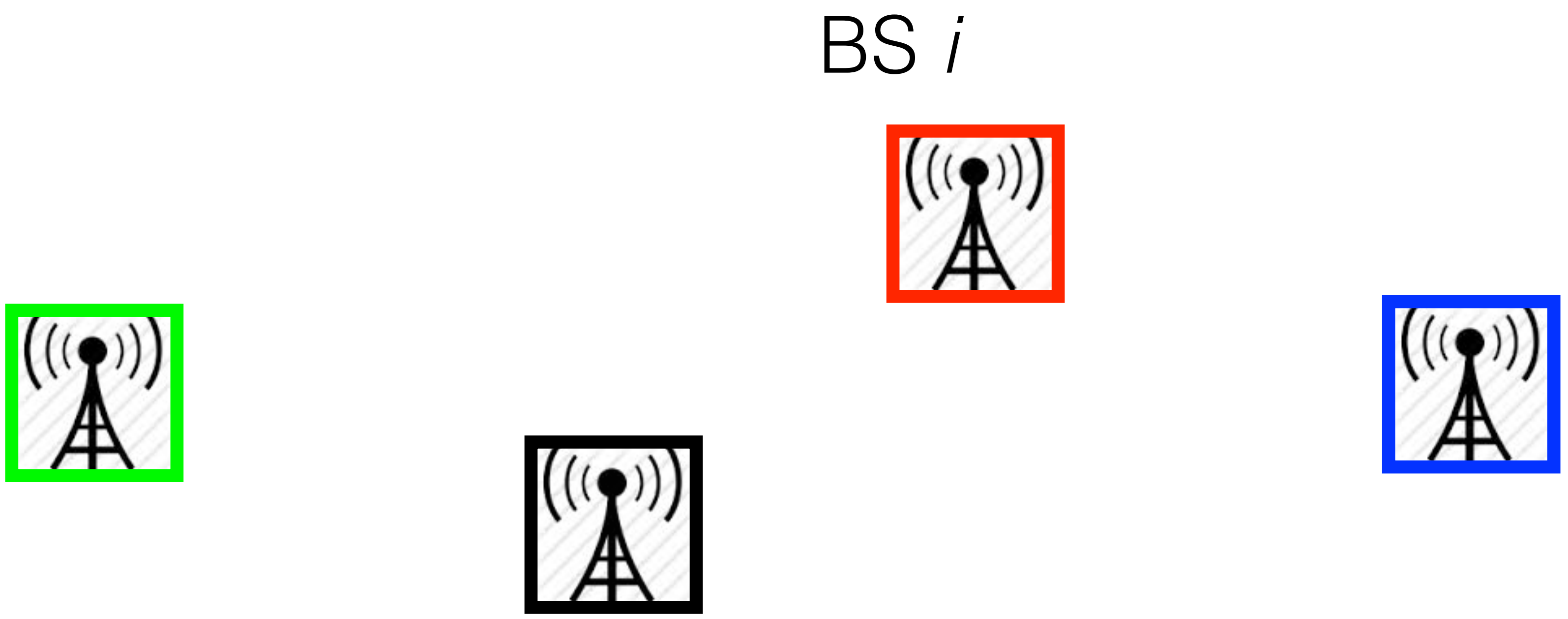
Again 1/2 Approx.

Online Downlink BS association

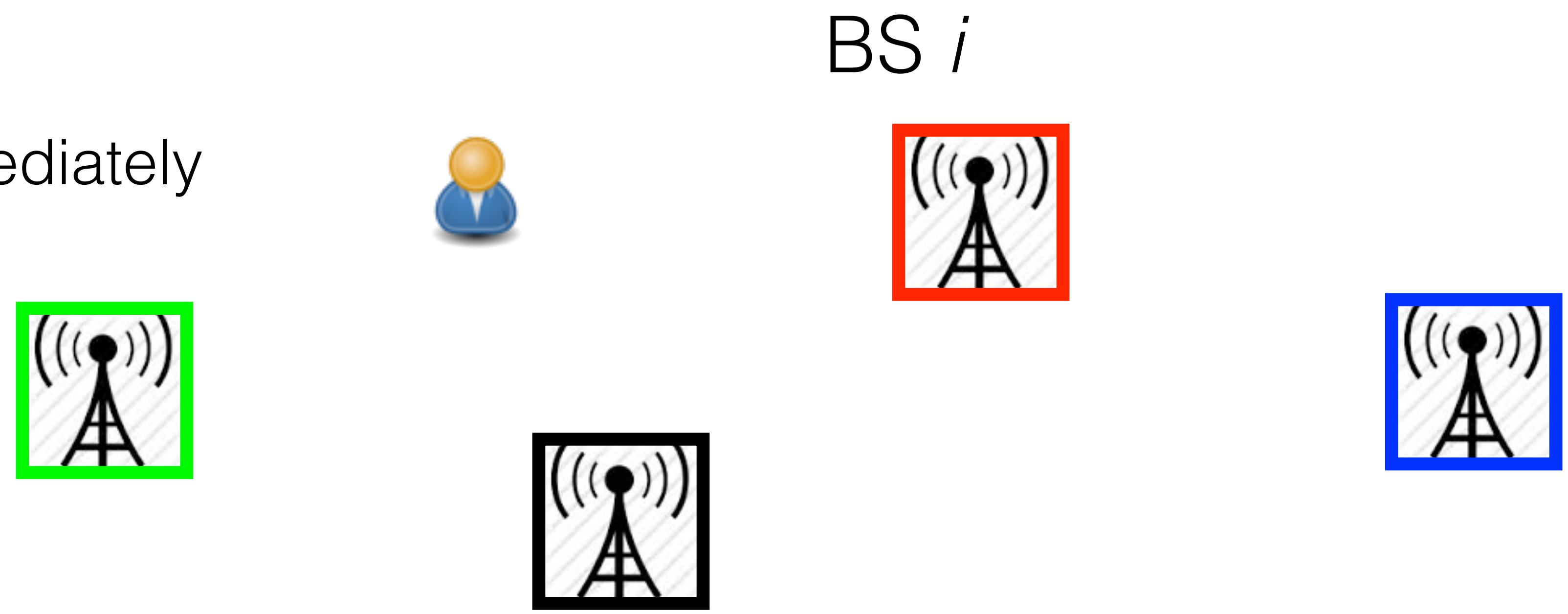


Disjoint bands for BS

Online Downlink BS association

users appear one by one

associate to one BS immediately



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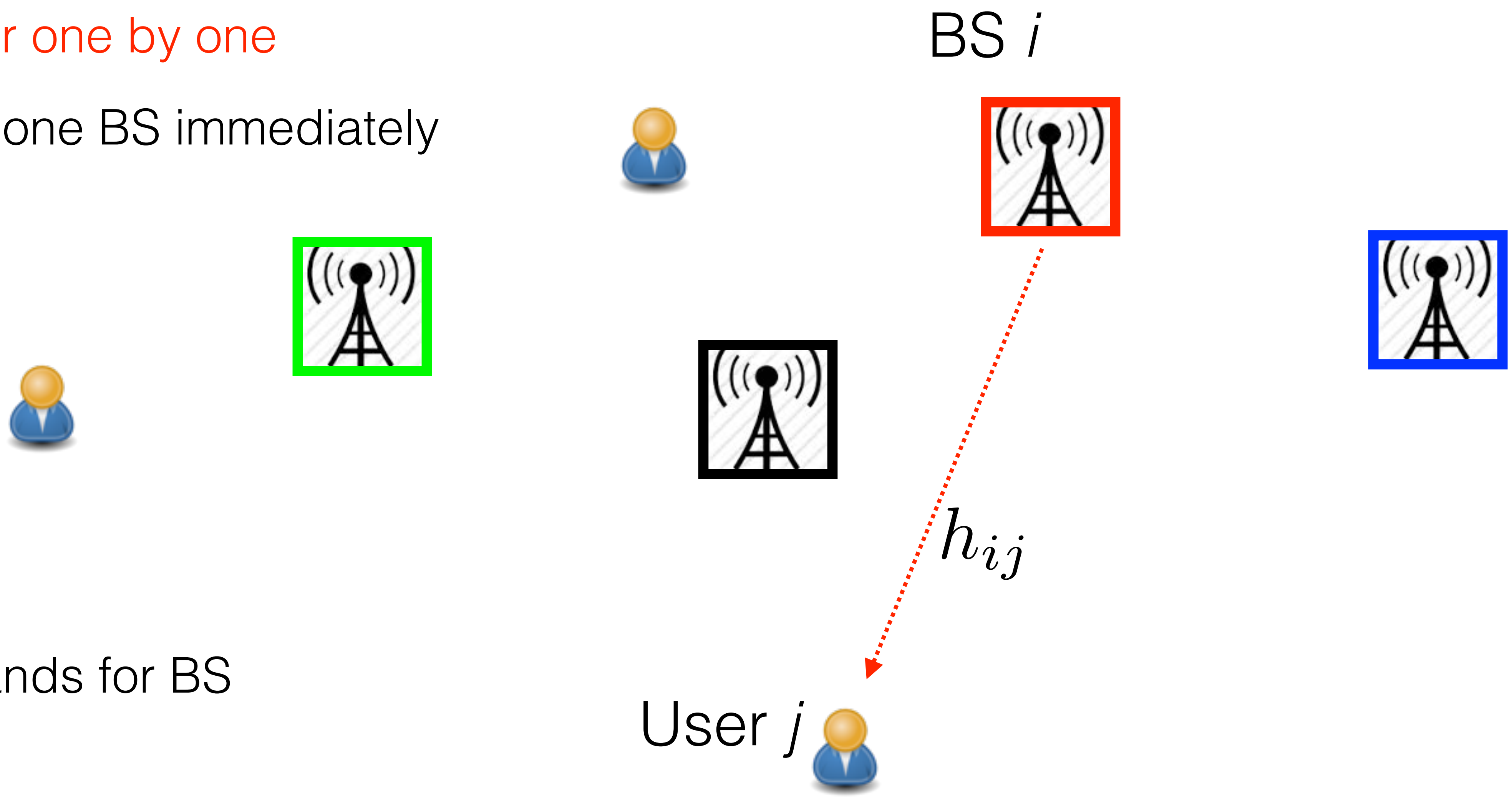

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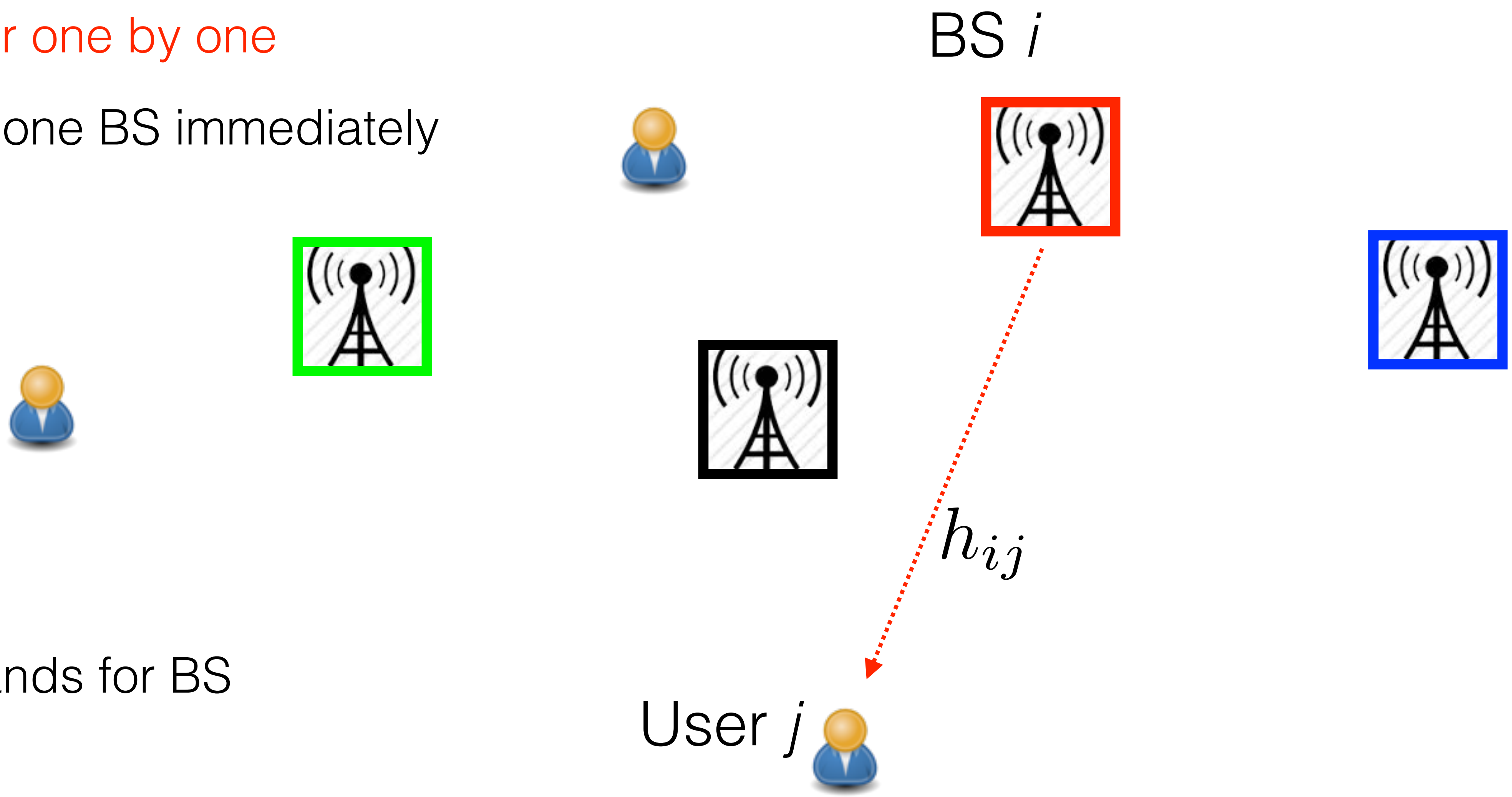


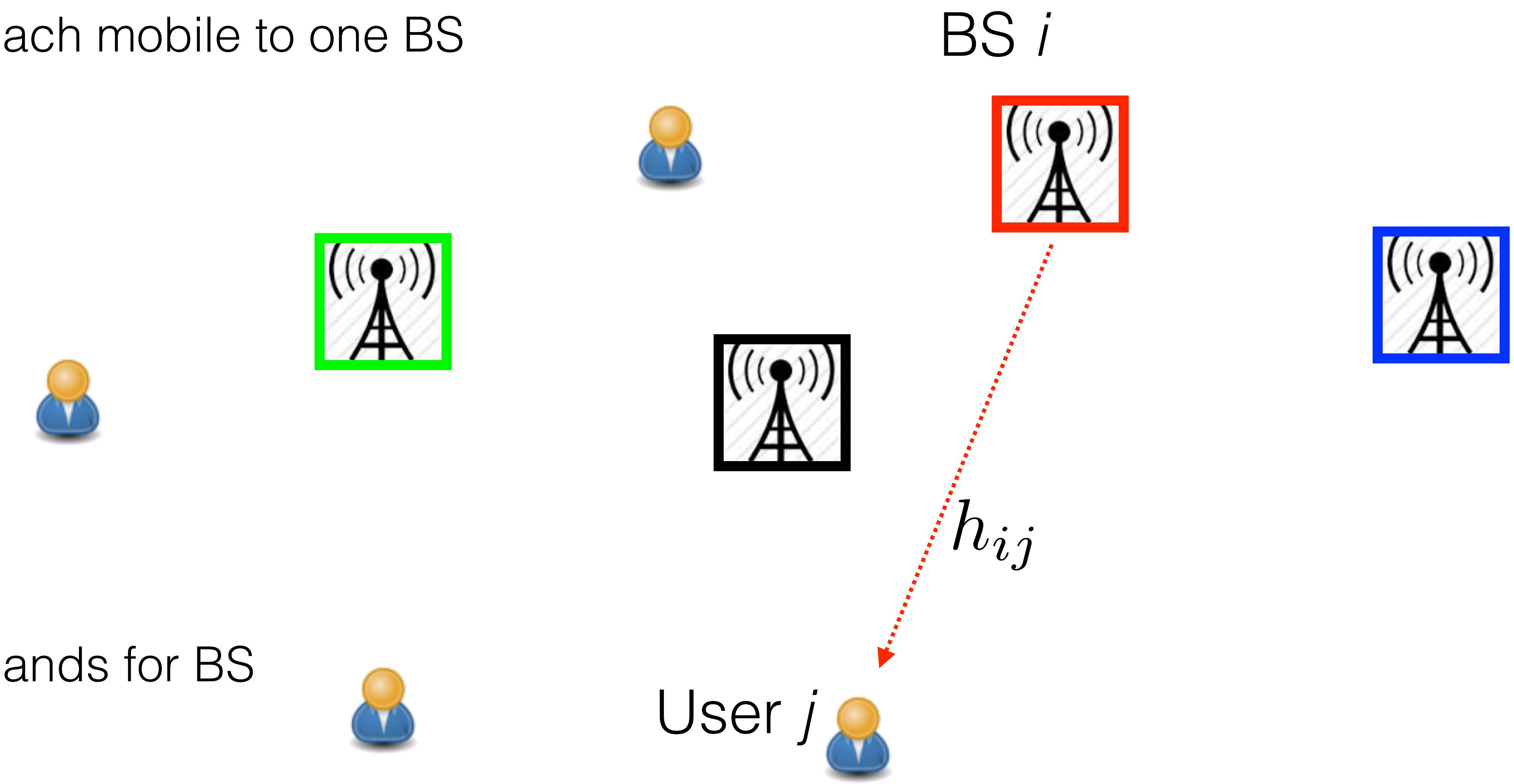
Disjoint bands for BS

Since the greedy algorithm works with one element (user) at a time

Again 1/2 Approx.

Untruthful Users Downlink BS association

Associate each mobile to one BS



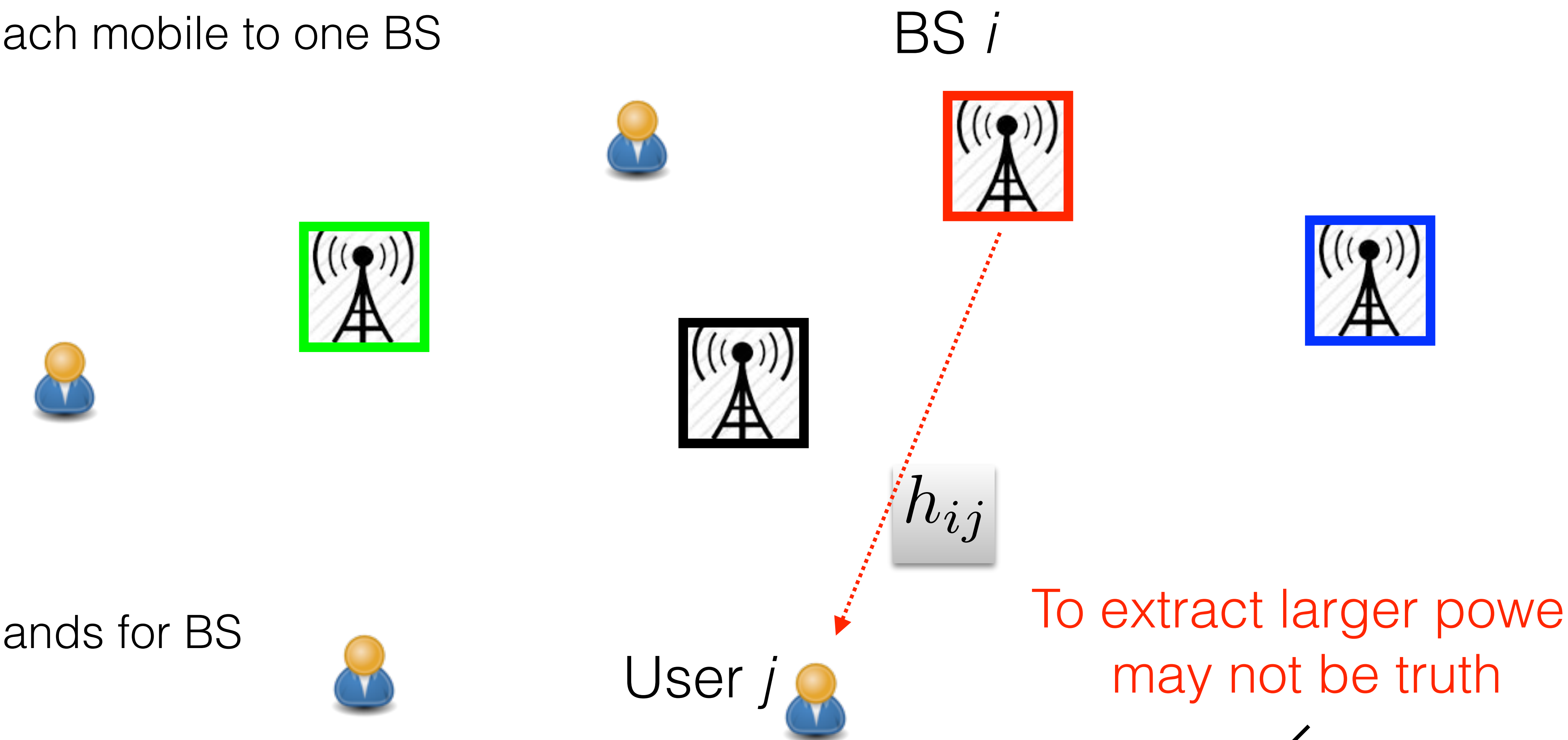
Disjoint bands for BS

Each BS gives power according to log utility for all associated users

$$R_i = \max_{\sum_{j \in A_i} P_j \leq P} \sum_{j \in A_i} \log(1 + P_j h_{ij})$$

Untruthful Users Downlink BS association

Associate each mobile to one BS



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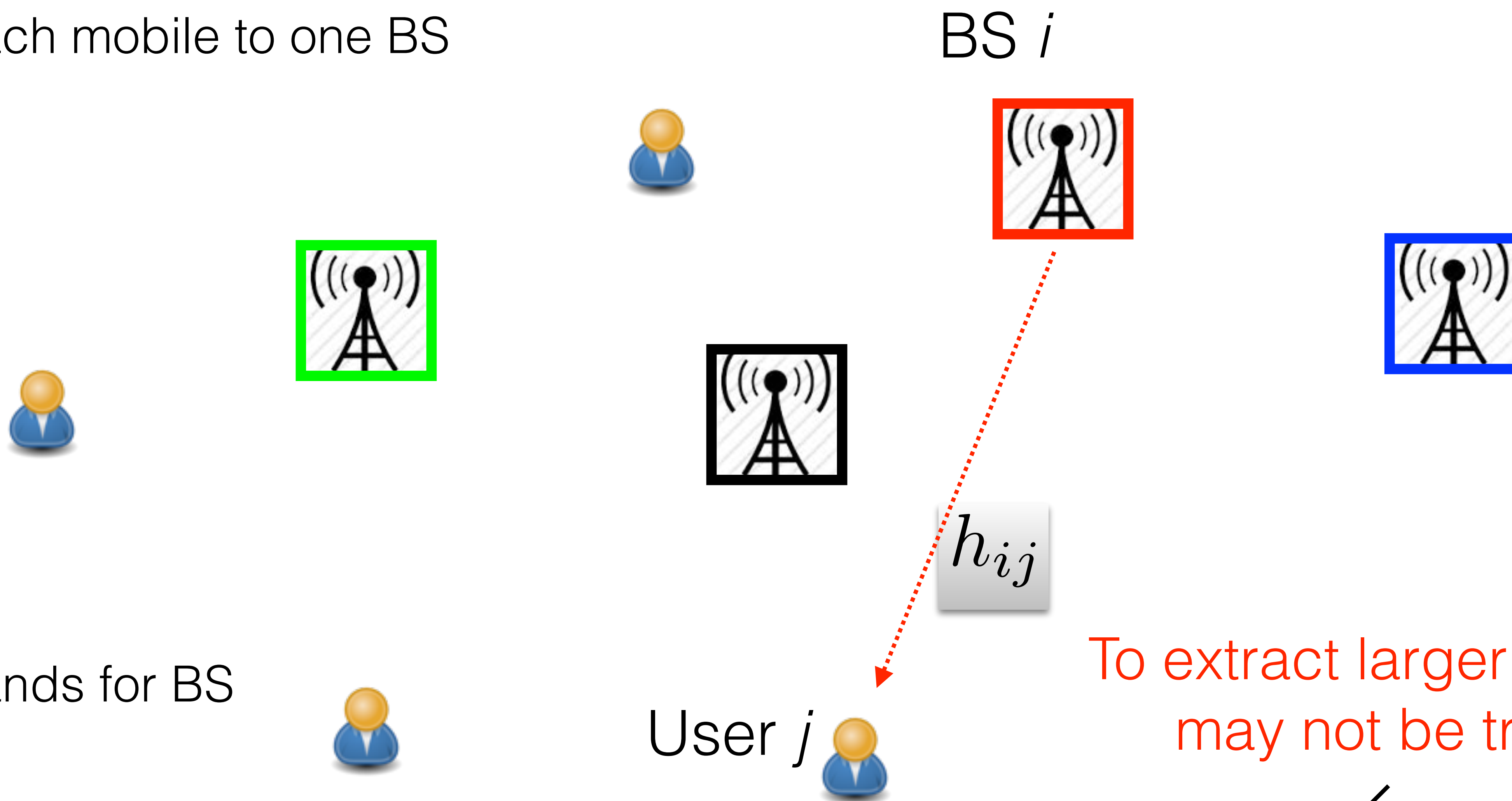

Disjoint bands for BS

To extract larger power may not be truth

Each BS gives power according to log utility for all associated users

$$R_i = \max_{\sum_{j \in A_i} P_j \leq P} \sum_{j \in A_i} \log(1 + P_j h_{ij})$$

Using VCG pricing again 1/2 Approximation

Lot more !